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Motivic rational homotopy type

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Abstract

We propose a motivic generalization of rational homotopy types. The algebraic invariants we study are defined as algebra objects in the category of mixed motives. This invariant plays a role of Sullivan's polynomial de Rham algebras. Another main notion is that of cotangent motives. Our main objective is to investigate the topological realization of these invariants and study their structures. Applying these machineries and the Tannakian theory, we construct actions of a derived motivic Galois group on rational homotopy types. Thanks to this, we deduce actions of the motivic Galois group of pro-unipotent completions of homotopy groups.

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1. Introduction

In this paper, we focus on motives for rational homotopy types of algebraic varieties. Rational homotopy theory originated from Quillen [43] and Sullivan [48]. Both approaches consider an algebraic invariant associated to a topological space that, under suitable conditions, encodes a rational homotopy type of the space. In Quillen's work, the algebraic invariant is a differential graded Lie algebra obtained from a simply connected topological space. In contrast, for a topological space S, Sullivan associated a commutative differential graded (dg) algebra $A_{PL}(S)$ of polynomial differential forms on S with rational coefficients. The cohomology ring of $A_{PL}(S)$ is isomorphic to the graded-commutative ring $H^*(S, \mathbb{Q})$ of the singular cohomology. In his approach, the main algebraic invariants of S are $A_{PL}(S)$ and its (so-called) Sullivan model.

We now turn our attention to algebraic varieties. A motivating source of motives is Hodge theory. From Morgan [39] and Hain [20], when S is a complex algebraic variety, a suitable model of $A_{PL}(S)$ admits a mixed Hodge structure in an appropriate setting. These works generalized the classical Hodge theory as Hodge theory for higher rational homotopy groups and unipotent fundamental groups, i.e., the pro-unipotent completion of the fundamental group. These additional structures have aguably been vital in algebraic geometry, such as the study of

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the topology of algebraic varieties. Meanwhile, in the 80's, a notion of motivic homotopy type was envisaged by Grothendieck in a footnote of Promenade 16 of [19]. Deligne and Goncharov developed a motivic theory for the pro-unipotent completions of fundamental groups in the setting of mixed Tate (and Artin-Tate) motives over a number field and its ring of integers [14].

Our investigation aims to define and study a motivic generalization of $A_{PL}(S)$. To gain some intuition for the invariants we will study, let us compare the homotopy (triangulated) category arising from topological spaces and the category of motives. Let $DM^{\otimes}(k)$ be the symmetric monoidal triangulated category of Voevodsky motives over a perfect field k, (here $DM^{\otimes}(k)$ admits infinite coproducts) [37], [50]. A pleasant feature of $DM^{\otimes}(k)$ is that its construction uses ideas from homotopy theory, allowing a clear analogy, and motivic cohomology groups appear as the hom sets in $DM(k)$. From the perspective of the analogy with homotopy theory, $DM^{\otimes}(k)$ is a motivic generalization of the homotopy category of module spectra over the Eilenberg-MacLane ring spectrum $H\mathbb{Z}$ (see [44] and [13, 14.2.9]). The motive $M(X) \in DM(k)$ associated to X [37] plays the role of the singular chain complex of a topological space. We now work with rational coefficients instead of $\mathbb{Z}$, and take the point of view that a topological counterpart of $DM^{\otimes}(k)$ is the derived category of $\mathbb{Q}$-vector spaces. Note that for a topological space S, $A_{PL}(S)$ is a commutative dg algebra with rational coefficients whereas the singular cochain complex $C^*(S,\mathbb{Q})$ is only a dg algebra that is not necessarily commutative. Thus, we consider that the commutative dg algebra $A_{PL}(S)$ amounts to the (underlying) complex $C^*(S,\mathbb{Q})$ endowed with an E_∞-algebra structure, that is, a commutative algebra structure in the operadic or $(\infty,1)$-categorical sense. This structure is crucial for rational homotopy theory. Further, the integral singular cochain complex $C^*(S,\mathbb{Z})$ admits an E_∞-algebra structure [5], [38], and is important for generalizations of rational homotopy theory such as integral homotopy theory [35]. To incorporate such structures, we must replace the derived category of $\mathbb{Q}$-vector spaces with its $(\infty,1)$-categorical enhancement, i.e., the derived $(\infty,1)$-category $\mathsf{D}(\mathbb{Q})$ of $\mathbb{Q}$-vector spaces that inherits a symmetric monoidal structure given by the tensor product of complexes. For an introduction to the $(\infty,1)$-categorical language, refer to [32, Chapter 1], [6] for example. Then, $A_{PL}(S)$ may be viewed as a commutative algebra object of the symmetric monoidal $(\infty,1)$-category $\mathsf{D}^{\otimes}(\mathbb{Q})$ in the $(\infty,1)$-categorical sense.

Let $\mathsf{DM}^{\otimes}(k)$ be a symmetric monoidal $(\infty,1)$-category of motives, that is, an $(\infty,1)$-categorical enhancement of $DM^{\otimes}(k)$. Let $\mathrm{CAlg}(\mathsf{DM}^{\otimes}(k))$ be the $(\infty,1)$-category of commutative algebra objects of $\mathsf{DM}^{\otimes}(k)$. From the above comparison, it is natural to think of a motivic generalization of $A_{PL}(-)$ as an object of $\mathrm{CAlg}(\mathsf{DM}^{\otimes}(k))$ whose underlying object in $\mathsf{DM}(k)$ is equivalent to the weak dual[1] of $M(X)$. We begin by associating an appropriate object of $\mathrm{CAlg}(\mathsf{DM}^{\otimes}(k))$ with a variety. There are (at least) two approaches to this:

(i) Let Sm_k denote the category of smooth schemes over k. This is equipped with the symmetric monoidal structure given by the product $X \times_k Y$. An object X of Sm_k can be viewed as a cocommutative coalgebra object such that the comultiplication is the diagonal $X \to X \times_k X$, and the counit is the structure morphism $X \to \operatorname{Spec} k$. If we regard the assignment $X \mapsto M(X)$ as a symmetric monoidal functor $\mathrm{Sm}_k \to \mathsf{DM}^{\otimes}(k)$, then $M(X)$ is a cocommutative coalgebra object in $\mathsf{DM}(k)$. Let $\mathbf{1}_k$ be a unit object in $\mathsf{DM}(k)$. Then the internal hom object $\mathsf{Hom}_{\mathsf{DM}(k)}(M(X), \mathbf{1}_k)$ inherits a commutative algebra structure in the $(\infty,1)$-categorical sense (i.e., an E_∞-algebra structure) from $M(X)$.

[1] By the weak dual of $M(X)$ we mean the internal hom object $\mathsf{Hom}_{\mathsf{DM}(k)}(M(X), \mathbf{1}_k)$ where $\mathbf{1}_k$ is a unit object of $\mathsf{DM}(k)$.

(ii) Let X be an object of Sm_k and let $f : X \to \operatorname{Spec} k$ be the structure morphism. Suppose a symmetric monoidal $(\infty, 1)$-category $\mathsf{DM}^{\otimes}(X)$ of motives over X is available, and there is an adjoint pair $f^* : \mathsf{DM}(k) \rightleftarrows \mathsf{DM}(X) : f_*$. If f^* is symmetric monoidal, then the right adjoint f_* is a lax symmetric monoidal functor, so that f_* sends a commutative algebra object in $\mathsf{DM}(X)$ to a commutative algebra object in $\mathsf{DM}(k)$. We denote by $\mathbf{1}_X$ a unit object of $\mathsf{DM}(X)$ and think of it as a commutative algebra object. We then have a commutative algebra object $f_*(\mathbf{1}_X)$, which is a natural candidate.

Approach (i) is reminiscent of the setup in topology regarding the relationship between singular chain complexes and singular cochain complexes, although the assignment $S \mapsto C_*(S, \mathbb{Z})$ is only oplax monoidal. We adopt approach (ii) since it gives a clear relationship with the relative situation. We use the formalism of motives over X, extensively developed by Cisinski and Déglise [13]. For a smooth scheme X, we define an object M_X of $\mathrm{CAlg}(\mathsf{DM}^{\otimes}(k))$, which we refer to as the cohomological motivic algebra of X (detailed definition in Section 3). In Section 3, we work with not only rational coefficients but an arbitrary coefficient ring.

The first fundamental property of M_X is that a (topological) realization of M_X is identified with the commutative dg algebra $A_{PL}(X^t)$ of polynomial differential forms on the underlying topological space X^t of $X \times_k \operatorname{Spec} \mathbb{C}$ when $k \subset \mathbb{C}$ (see Theorem 4.3 in Section 4). To Weil cohomology theory such as singular cohomology, analytic/algebraic de Rham cohomology, l-adic étale cohomology, one can associate a symmetric monoidal functor called a realization functor:

$$\mathsf{R} : \mathsf{DM}^{\otimes}(k) \to \mathsf{D}^{\otimes}(K)$$

where K is a coefficient field of cohomology theory, and $\mathsf{D}^{\otimes}(K)$ is the symmetric monoidal derived $(\infty, 1)$-category of K-vector spaces. The field K is assumed to be of characteristic zero. For example, when k is embedded in $\mathbb{C}$, the realization functor $\mathsf{R} : \mathsf{DM}^{\otimes}(k) \to \mathsf{D}^{\otimes}(\mathbb{Q})$ associated to singular cohomology theory (with rational coefficients) carries $M(X)$ to a complex quasi-isomorphic to the singular chain complex $C_*(X^t, \mathbb{Q})$ of the underlying topological space X^t. Notice that the realization functor is symmetric monoidal. It gives rise to a functor

$$\mathrm{CAlg}(\mathsf{DM}^{\otimes}(k)) \to \mathrm{CAlg}(\mathsf{D}^{\otimes}(K)),$$

which we call the multiplicative realization functor, where $\mathrm{CAlg}(\mathsf{D}^{\otimes}(K))$ is the $(\infty, 1)$-category of commutative algebra objects in $\mathsf{D}^{\otimes}(K)$. One can naturally identify $\mathrm{CAlg}(\mathsf{D}^{\otimes}(K))$ with the $(\infty, 1)$-category obtained from the category of commutative dg algebras over K by inverting quasi-isomorphisms (see Section 2). In the case of singular cohomology, we have $\mathrm{CAlg}(\mathsf{DM}^{\otimes}(k)) \to \mathrm{CAlg}(\mathsf{D}^{\otimes}(\mathbb{Q}))$. The commutative dg algebra $A_{PL}(X^t)$ appears as the image of M_X under the multiplicative realization functor. Namely, in addition to being analogous, the multiplicative realization functor relates M_X with $A_{PL}(X^t)$. Importantly, this allows many operations on $A_{PL}(X^t)$ to be promoted to a motivic level. For example, the multiplicative realization functor preserves (small) colimits. Suppose that x is a k-rational point on X. Let $\epsilon : A_{PL}(X^t) \to \mathbb{Q}$ be the augmentation induced by the point x on X^t. The bar construction of the augmented commutative dg algebra can be described in terms of colimits (see e.g. [41, 4.7], [51, Section 3], [25]). Thus, it is possible to promote the bar construction of $A_{PL}(X^t) \to \mathbb{Q}$ to a bar construction of $M_X \to \mathbf{1}_k$ in $\mathrm{CAlg}(\mathsf{DM}^{\otimes}(k))$.

Cotangent motives. By using M_X we introduce a new invariant of a pointed smooth scheme (X, x) over a perfect field, which lies in $\mathsf{DM}(k)$ (see Section 6). The invariant $LM_{(X,x)}$ in $\mathsf{DM}(k)$ is defined by means of the cotangent complex of M_X. Thanks to the foundational work of Lurie on cotangent complexes in a very general setting, we are able to apply it to the motivic situation. We shall call $LM_{(X,x)}$ the cotangent motive of X at x (cf. Definition 6.1). The remarkable features of cotangent motives include:

(i) $LM_{(X,x)}$ may be viewed as a motive for the duals of rational homotopy groups (in the case of the simply connected varieties).

(ii) If (X,x) is a semi-abelian variety with the origin, then $LM_{(X,x)}$ is equivalent to the dual of the 1-motive of X. Therefore, the notion of cotangent motives is also a generalization of 1-motives.

(iii) There is a natural map from Voevodsky's motive, namely, the dual Hurewicz map $M(X)^\vee \to \mathbf{1}_k \oplus LM_{(X,x)}$.

We prove that the rational homotopy groups appear as the realization of $LM_{(X,x)}$ (see Theorems 6.4 and 6.11 in Section 6). Namely, when k is embedded in $\mathbb{C}$ and the underlying topological space X^t is simply connected, $H^i(\mathsf{R}(LM_{(X,x)}))$ is the dual of the i-th rational homotopy group of X^t. In addition, $H^1(\mathsf{R}(LM_{(X,x)}))$ can be identified with the cotangent space of the origin of the pro-unipotent completion of the fundamental group, that is, the "linear data" of the fundamental group.

Although $LM_{(X,x)}$ has less information than M_X, the cotangent motive $LM_{(X,x)}$ has a more direct relation with homotopy groups than M_X. We apply an explicit computational study of M_X in Section 5 to compute $LM_{(X,x)}$. For example, as mentioned above, recall that if (X,x) is a semi-abelian variety with the origin, then $LM_{(X,x)}$ is a dual of the 1-motive of X. If $\mathbb{P}^n$ is the n-dimensional projective space (over a perfect field) endowed with a base point x, then

$$LM_{(\mathbb{P}^n,x)} \simeq \mathbf{1}_k(-1)[-2] \oplus \mathbf{1}_k(-n-1)[-2n-1],$$

where "(s)" and "$[t]$" indicate the Tate twist and the shift, respectively. This means that $\mathbf{1}_k(1)$ is a "motive for the second rational homotopy group", and $\mathbf{1}_k(n+1)$ is a "motive for the $(2n+1)$-th rational homotopy group" (see Remark 6.14 in Section 6).

Structure of cohomological motivic algebras. To explicitly understand cohomological motivic algebras and cotangent motives, it is natural to consider explicit structures on M_X. In Section 5, as a first step towards the computational study, we describe an explicit structure of the cohomological motivic algebra for some cases. For this purpose, we adopt an approach that traces back to Sullivan's work. A (minimal) Sullivan model of $A_{PL}(S)$ is given by an iterated homotopy pushout of free commutative dg algebras (see e.g. [21], [22], [16] or the beginning of Section 5). We apply this to several cases such as projective spaces to explicitly describe M_X as a colimit of a similar diagram of free commutative algebra objects in $\mathrm{CAlg}(\mathsf{DM}^\otimes(k))$. Unlike classical rational homotopy theory, the study of M_X is not so simple even in relatively elementary cases; we need some devices and deep results. This difference may be regarded as a reflection of the fact that M_X has rich structures. For instance, let us consider a proper smooth curve C of genus $g > 1$ with a base k-rational point c. Let J_C be the Jacobian variety and let $u : C \to J_C$ be the Abel-Jacobi morphism. We here take a viewpoint that the Abel-Jacobi morphism is an "algebraic abelianization" of C: when $k = \mathbb{C}$, the map $C^t \to J_C^t$ of the underlying toplogical spaces induces an abelianization $\pi_1(C^t, c) \to \pi_1(J_C^t, u(c)) \simeq \pi_1(C^t, c)^{ab}$. The Abel-Jacobi morphism u induces a morphism $u^* : M_{J_C} \to M_C$ of cohomological motivic algebras, which gives rise to an inductive sequence in $\mathrm{CAlg}(\mathsf{DM}^\otimes(k))$:

$$M_{J_C} = M_1 \to M_2 \to \cdots \to M_n \to M_{n+1} \to \cdots \to M_C$$

that decomposes $u^* : M_{J_C} \to M_C$ such that M_C is a filtered colimit $\varinjlim_{n\geq 1} M_n$ (see Sections 5.1.5 and 5.3.1). This sequence (or co-tower) starting with M_{J_C} can be thought of as a structure of M_C or a refined Abel-Jacobi morphism. It is notable that it does not exist in the category of schemes and does not arise from $\mathsf{DM}(k)$. Roughly speaking, this sequence gives a step-by-step description of the non-abelian nature of C that starts with its "abelian part" M_{J_C}, and each $M_n \to M_{n+1}$ contains a motivic structure of the (unipotent) fundamental group. From a perspective of the formality, it is not generally reasonable to expect a formality of M_X of a smooth projective variety X even if one can define a formality by using a motivic t-structure. Assume that there exist a motivic t-structure on $\mathsf{DM}(k)$ (or on the full subcategory of compact objects) and the associated heart $\mathcal{MM} \subset \mathsf{DM}(k)$ (see e.g., [24, Section 1] and references therein for the notion of motivic t-structures). We then have the symmetric monoidal derived ∞-category $\mathsf{D}(\mathcal{MM})$ of $\mathcal{MM}$ and a (essentially unique) symmetric monoidal functor $\mathsf{D}(\mathcal{MM}) \to \mathsf{DM}(k)$ which extends the inclusion $\mathcal{MM} \hookrightarrow \mathsf{DM}(k)$. Let $H(M_X)$ be a graded commutative algebra obtained from M_X by passing to cohomology with respect to the motivic t-structure. In principle, we should say that M_X is formal when M_X can be encoded by $H(M_X)$ (see also Remark 5.30). But it is not reasonable to hope that the image of $H(M_X)$ under $\mathrm{CAlg}(\mathsf{D}(\mathcal{MM})) \to \mathrm{CAlg}(\mathsf{DM}(k))$ is equivalent to M_X. Indeed, there is a counterexample to the formality at the Hodge level (see [10]).

Tannakian aspect. In Sections 7 and 8, we discuss Tannakian presentation of motivic structures on rational homotopy invariants. In Section 7, we construct so-called motivic Galois actions on $A_{PL}(-)$ of the underlying space and the related topological invariants. Recall that various "topological invariants" of algebraic varieties are equipped with actions of groups. For example, l-adic étale cohomology groups admit actions of the absolute Galois group, and a Hodge structure can be described by an action of the Mumford-Tate group. In the motivic setting, the group should be a motivic Galois group (for a comprehensive account of the motivic perspective, refer to [3] for example). In our context, we adopt the derived motivic Galois group MG introduced in [24], and the associated pro-algebraic group MG which we call the motivic Galois group (see the beginning of Section 7, Section 7.3, and [24]). From the conventional Tannakian viewpoint, motivic structures are the suitably completed homotopy groups endowed with actions of the motivic Galois group. In an appropriate setting (including $k \subset \mathbb{C}$), we construct (cf. Corollary 7.6, Theorem 7.17, Corollary 7.18)

- a canonical action of MG on $A_{PL}(X^t)$,
- a canonical action of MG on the pro-unipotent completion $\pi_1(X^t, x)_{uni}$ of the fundamental group,
- a canonical action of MG on $\pi_i(X^t, x)_{uni}$ $(i > 1)$ when X is nilpotent and of finite type.

The second and third actions are induced by the first one, that is, the "enhanced action" of MG on $A_{PL}(X^t)$. In order to obtain the action of MG on $A_{PL}(X^t)$ we apply the results in Section 4 and the Tannakian formalism developed in [24]. Even if one is ultimately interested in the actions on completed homotopy groups, the enhanced action on $A_{PL}(X^t)$ performs a pivotal role in the construction.

Homotopy exact sequence. Recall the homotopy exact sequence for étale fundamental groups

$$1 \to \pi_1^{\text{ét}}(X \times_k \operatorname{Spec} \bar{k}, \bar{x}) \to \pi_1^{\text{ét}}(X, \bar{x}) \to \mathrm{Gal}(\bar{k}/k) \to 1$$

where $\bar{k}$ is a separable closure of k. In Section 8, by means of the Tannakian theory developed in [26], when X is an algebraic curve we formulate and prove a version of the homotopy exact sequence with the derived motivic Galois group (or stack) instead of $\mathrm{Gal}(\bar{k}/k)$ (see Proposition 8.12).

In the Appendix, for the case of mixed Tate motives over a number field, we compare our approach and an approach to motivic fundamental groups by Deligne and Goncharov. We hope that the comparison is helpful for understanding the concepts and the connections between them.

In this paper, we mainly focus on the singular (or Betti) realization. Although we did not focus on other realizations, such as étale, de Rham, Hodge or crystalline realizations, our results would also be useful for other realizations, and the study of various realizations of M_X and $LM_{(X,x)}$ is expected to be more fruitful. For example, if one considers the Hodge realization functor in the ∞-categorical setting, it immediately yields a new conceptual construction of Hodge structures on $A_{PL}(-)$ and rational homotopy groups of the underlying topological space, at least in the case of simply connected varieties.

Outline of the paper. Section 2 collects some preliminaries on ∞-categories and clarifies the terminology used in this paper. Section 3 introduces the cohomological motivic algebras of smooth schemes and their variants. The definition is formal. Our principle is that the cohomological motivic algebra M_X is an algebraic invariant that represents the "motivic rational homotopy type" of X. In Section 4, we start with the review of the the commutative differential graded algebra $A_{PL}(S)$ of polynomial differential forms. We prove that the singular realization of the cohomological motivic algebra M_X is equivalent to $A_{PL}(X^t)$. In Section 5, we give explicit structures of M_X for several cases. These computations are used in Section 6, which introduces the notion of cotangent motives. We describe the realizations of cotangent motives as the (duals of) rational homotopy groups. Moreover, we give explicit computations for some cases. In Sections 7 and 8, we develop a Tannakian theory of motivic rational homotopy types. In Section 7 we construct actions of the motivic Galois group on the unipotent completion of homotopy groups. In Section 8 we give a Tannakian presentation of cohomological motivic algebras in the case of algebraic curves. We prove an analog of the homotopy exact sequence. To this end, we make use of the theory of fine Tannakian ∞-categories developed in [26]. In the Appendix, we prove a comparison theorem that describes the relationship between Deligne-Gocharov's work and our work.

2. Notation and Convention

2.1 We shall use the theory of *quasi-categories* extensively developed by Joyal and Lurie from the viewpoint of $(\infty, 1)$-categories. This theory provides us with powerful tools and adequate language for our purpose, though a part of contents might be reformulated in term of other languages such as model categories or the likes. Following [32], we shall refer to quasi-categories as ∞-*categories*. Our main references are [32] and [33]. We assume that the reader is familiar with ∞-categories. To an ordinary category $\mathcal{C}$, one can assign an ∞-category by taking its nerve $\mathrm{N}(\mathcal{C})$. Such simplicial sets $\mathrm{N}(\mathcal{C})$ arising from ordinary categories naturally constitute a full subcategory of the simplicial category of ∞-categories. Therefore, when we treat ordinary categories we often omit the nerve $\mathrm{N}(-)$ and think of them directly as ∞-categories. We often refer to a map $S \to T$ of ∞-categories as a functor. We call a vertex in an ∞-category S (resp. an edge) an object (resp. a morphism). We use Grothendieck universes $\mathbb{U} \in \mathbb{V} \in \mathbb{W} \in \dots$ and usual mathematical objects such as groups, rings, vector spaces are assumed to belong to $\mathbb{U}$.

Here is a list of (some) of the convention and notation that we will use.

- Δ: the category of linearly ordered finite sets (consisting of $[0], [1], \ldots, [n] = \{0, \ldots, n\}, \ldots$)
- Δ^n: the standard n-simplex as the simplicial set represented by $[n]$,
- Set_Δ: the category of simplicial sets,
- N: the simplicial nerve functor (cf. [32, 1.1.5])
- Γ: the nerve of the category of pointed finite sets, $\langle 0\rangle = \{*\}, \langle 1\rangle = \{*, 1\}, \ldots, \langle n\rangle = \{*, 1, \ldots, n\}$,
- $\mathcal{C}^{op}$: the opposite ∞-category of an ∞-category $\mathcal{C}$. For a functor $F : \mathcal{C} \to \mathcal{D}$, we denote by $F^{op} : \mathcal{C}^{op} \to \mathcal{D}^{op}$ the induced functor.
- Let $\mathcal{C}$ be an ∞-category and suppose that we are given an object c. Then $\mathcal{C}_{c/}$ and $\mathcal{C}_{/c}$ denote the undercategory and overcategory, respectively (cf. [32, 1.2.9]).
- $\mathcal{C}^{\simeq}$: the largest Kan subcomplex (contained) in an ∞-category $\mathcal{C}$, that is, the Kan complex obtained from $\mathcal{C}$ by restricting to those morphisms (edges) which are equivalences.
- Cat_∞: the ∞-category of small ∞-categories, Similarly, $\widehat{\mathrm{Cat}}_\infty$ denotes the ∞-category of large ∞-categories (i.e., ∞-categories that belong to $\mathbb{V}$),
- $\mathcal{S}$: ∞-category of small spaces. We denote by $\widehat{\mathcal{S}}$ the ∞-category of large ∞-spaces (cf. [32, 1.2.16])
- $\mathrm{h}(\mathcal{C})$: homotopy category of an ∞-category (cf. [32, 1.2.3.1])
- $\mathrm{Fun}(A, B)$: the function complex for simplicial sets A and B
- $\mathrm{Fun}_C(A, B)$: the simplicial subset of $\mathrm{Fun}(A, B)$ classifying maps which are compatible with given projections $A \to C$ and $B \to C$.
- $\mathrm{Map}(A, B)$: the largest Kan subcomplex of $\mathrm{Fun}(A, B)$ when B is an ∞-category.
- $\mathrm{Map}_\mathcal{C}(C, C')$: the mapping space from an object $C \in \mathcal{C}$ to $C' \in \mathcal{C}$ where $\mathcal{C}$ is an ∞-category. We usually view it as an object in $\mathcal{S}$ (cf. [32, 1.2.2]). If $\mathcal{C}$ is an ordinary category, we write $\mathrm{Hom}_\mathcal{C}(C, C')$ for the hom set.
- $C^\vee$: For an object C of a symmetric monoidal ∞-category $\mathcal{C}$, we write $C^\vee$ for a dual of C when C is a dualizable object. If there are internal objects, we also write $C^\vee$ also for the weak dual, that is, the internal hom object $\mathsf{Hom}_\mathcal{C}(C, \mathbf{1}_\mathcal{C})$ with $\mathbf{1}_\mathcal{C}$ a unit object.
- $\mathrm{Ind}(\mathcal{C})$: ∞-category of Ind-objects in an ∞-category $\mathcal{C}$ (see [32, 5.3.5.1], [33, 4.8.1.14] for the symmetric monoidal setting).
- Pr^L: the ∞-category of presentable ∞-categories whose morphisms are left adjoint functors.

2.2 *From model categories to ∞-categories.* We recall Lurie's construction by which one can obtain ∞-categories from a category (more generally ∞-category) endowed with a prescribed collection of morphisms (see [33, 1.3.4, 4.1.7, 4.1.8] for details). It can be viewed as an alternative approach to Dwyer-Kan hammock localization. Let $\mathcal{D}$ be a category and let W be a collection of morphisms in $\mathcal{D}$ which is closed under composition and contains all isomorphisms. A typical example of $(\mathcal{D}, W)$ which we have in mind is $(\mathbb{M}, W_\mathbb{M})$ such that $\mathbb{M}$ is a model category (see e.g. [32, Appendix], [23]) and $W_\mathbb{M}$ is the collection of all weak equivalences. For $(\mathcal{D}, W)$, there is an ∞-category $\mathrm{N}(\mathcal{D})[W^{-1}]$ and a functor $\xi : \mathrm{N}(\mathcal{D}) \to \mathrm{N}(\mathcal{D})[W^{-1}]$ such that for any ∞-category $\mathcal{C}$ the composition induces a fully faithful functor

$$\mathrm{Map}(\mathrm{N}(\mathcal{D})[W^{-1}], \mathcal{C}) \to \mathrm{Map}(\mathrm{N}(\mathcal{D}), \mathcal{C})$$

whose essential image consists of those functors $F : \mathrm{N}(\mathcal{D}) \to \mathcal{C}$ such that F carry morphisms lying in W to equivalences in $\mathcal{C}$. We shall refer to $\mathrm{N}(\mathcal{D})[W^{-1}]$ as the ∞-category obtained from $\mathcal{D}$ by inverting morphisms in W. Consider $(\mathbb{M}, W_{\mathbb{M}})$ such that $\mathbb{M}$ is a combinatorial model category and $W_{\mathbb{M}}$ is the collection of weak equivalences. The ∞-category $\mathbb{M}^c[W^{-1}] := \mathrm{N}(\mathbb{M}^c)[(\mathbb{M}^c \cap W_{\mathbb{M}})^{-1}]$ is presentable where $\mathbb{M}^c$ is the full subcategory of cofibrant objects. (When $\mathbb{M}$ is a monoidal model category, it is convenient to work with the full subcategory of cofibrant objects $\mathbb{M}^c \subset \mathbb{M}$ instead of $\mathbb{M}$.) If $\mathbb{M}$ is a stable model category, then $\mathbb{M}^c[W^{-1}]$ is a stable ∞-category (cf. [24]). The homotopy category of $\mathbb{M}^c[W^{-1}]$ coincides with the homotopy category of the model category $\mathbb{M}$. If $\mathbb{M}$ is a symmetric monoidal model category (whose unit object is cofibrant), $\mathbb{M}^c[W^{-1}]$ is promoted to a symmetric monoidal ∞-category $\mathbb{M}^c[W^{-1}]^{\otimes} := \mathrm{N}(\mathbb{M}^c)[(\mathbb{M}^c \cap W_{\mathbb{M}})^{-1}]^{\otimes}$ (see below for symmetric monoidal ∞-categories). In addition, there is a symmetric monoidal functor $\tilde{\xi} : \mathrm{N}(\mathbb{M}^c)^{\otimes} \to \mathbb{M}^c[W^{-1}]^{\otimes}$ which has ξ as the underlying functor and satisfies a similar universal property. If $\mathbb{M}$ is combinatorial, then the tensor product $\otimes : \mathbb{M}^c[W^{-1}] \times \mathbb{M}^c[W^{-1}] \to \mathbb{M}^c[W^{-1}]$ preserves small colimits separately in each variable. Let $\mathbb{L}$ be another symmetric monoidal model category and let $\phi : \mathbb{M} \to \mathbb{L}$ be a symmetric monoidal functor. If ϕ carries cofibrant objects to cofibrant objects and preserves weak equivalences between them (e.g. symmetric monoidal left Quillen functors), it induces a symmetric monoidal functor $\mathbb{M}^c[W^{-1}]^{\otimes} \to \mathbb{L}^c[W^{-1}]^{\otimes}$ of symmetric monoidal ∞-categories.

2.3 *Symmetric monoidal ∞-categories, modules and algebras.* We use the theory of (symmetric) monoidal ∞-categories developed in [33]. A symmetric monoidal ∞-category is a coCartesian fibration $\mathcal{C}^{\otimes} \to \Gamma$ that satisfies a "symmetric monoidal condition", see [33, 2.1.2]. For a symmetric monoidal ∞-category $\mathcal{C}^{\otimes} \to \Gamma$, we often write $\mathcal{C}$ for the underlying ∞-category. Also, by abuse of notation, we usually use the superscript in $\mathcal{C}^{\otimes}$ to indicate a symmetric monoidal structure on an ∞-category. For a symmetric monoidal ∞-category $\mathcal{C}^{\otimes}$, we write $\mathrm{CAlg}(\mathcal{C}^{\otimes})$ (or simply $\mathrm{CAlg}(\mathcal{C})$) for the ∞-category of commutative algebra objects in $\mathcal{C}^{\otimes}$. Let A be a commutative ring spectrum, that is, a commutative algebra object in the category Sp of spectra. We write $\mathrm{Mod}_A^{\otimes}$ for the symmetric monoidal ∞-category of A-module spectra, (see e.g. [33]). We put $\mathrm{CAlg}_A = \mathrm{CAlg}(\mathrm{Mod}_A^{\otimes})$. For an ordinary commutative ring K, we put $\mathrm{Mod}_K^{\otimes} := \mathrm{Mod}_{HK}^{\otimes}$ and $\mathrm{CAlg}_K := \mathrm{CAlg}_{HK}$ where HK is the Eilenberg-MacLane ring spectrum.

Let K be a field of characteristic zero. Let $\mathrm{Comp}^{\otimes}(K)$ be the symmetric monoidal category of cochain complexes of K-vector spaces (the symmetric monoidal structure is given by the tensor product of cochain complexes). This category admits a projective combinatorial symmetric monoidal model structure, whose weak equivalences are quasi-isomorpisms, and whose cofibrations (resp. fibrations) are degreewise monomorphisms (resp. epimorphisms), see e.g. [23, Section 2.3] or [33, 7.1.2.11]. We shall write $\mathsf{D}^{\otimes}(K)$ for the symmetric monoidal stable presentable ∞-category obtained from $\mathrm{Comp}^{\otimes}(K)$ by inverting weak equivalences. According to [33, 7.1.2.12, 7.1.2.13], there is a canonical equivalence $\mathsf{D}^{\otimes}(K) \simeq \mathrm{Mod}_K^{\otimes}$. We refer to $\mathsf{D}^{\otimes}(K)$ and $\mathrm{Mod}_K^{\otimes}$ as the (symmetric monoidal) derived ∞-category of K-vector spaces. The equivalence $\mathsf{D}^{\otimes}(K) \simeq \mathrm{Mod}_K^{\otimes}$ induces $\mathrm{CAlg}(\mathsf{D}^{\otimes}(K)) \simeq \mathrm{CAlg}_K = \mathrm{CAlg}(\mathrm{Mod}_K^{\otimes})$. Let CAlg_K^{dg} be the category of commutative differential graded K-algebras. A commutative differential graded K-algebras is a commutative algebra object in $\mathrm{Comp}^{\otimes}(K)$. There is a natural forgetful functor $U : \mathrm{CAlg}_K^{dg} \to \mathrm{Comp}(K)$. The category CAlg_K^{dg} admits a combinatorial model structure such that a morphism f is a weak equivalences (resp. a fibration) if and only if $U(f)$ is a quasi-isomorphism (resp. a epimorphism) (here, we use the assumption of characteristic zero). If we write $\mathrm{N}(\mathrm{CAlg}_K^{dg})[W^{-1}]$ for the ∞-category obtained from CAlg_K^{dg} by inverting weak equivalences, then there is a canonical equivalences $\mathrm{N}(\mathrm{CAlg}_K^{dg})[W^{-1}] \simeq \mathrm{CAlg}_K$ (see [33, 7.1.4.10, 7.1.4.11], [33, 4.5.4.6]). We often use these equivalences

$$\mathrm{N}(\mathrm{CAlg}_K^{dg})[W^{-1}] \simeq \mathrm{CAlg}(\mathsf{D}^{\otimes}(K)) \simeq \mathrm{CAlg}_K\,.$$

A variety is a geometrically connected scheme separated of finite type over a field.

3. Cohomological motivic algebras

Let K be a commutative ring.

3.1 We work with ∞-categories of mixed motives. We begin by setting up ∞-categories of mixed motives. They are obtained from model (dg, etc) categories of motives or by the ∞-categorical version of Voevodsky's construction. In this paper, we adopt to use symmetric monoidal model categories constructed by Cisinski and Déglise [11], [13]. Let X be a smooth scheme separated of finite type over a perfect field k (or more generally, a noetherian regular scheme). Let Sm_X denote the category of smooth schemes separated of finite type over X. Let $\mathcal{N}^{tr}(X)$ be the Grothendieck abelian category of Nisnevich sheaves of K-modules with transfers over X (see e.g. [11, Example 2.4] or [13] for this notion). Let $\mathrm{Comp}(\mathcal{N}^{tr}(X))$ be the symmetric monoidal category of (possibly unbounded) cochain complexes of $\mathcal{N}^{tr}(X)$. Then $\mathrm{Comp}(\mathcal{N}^{tr}(X))$ admits a stable symmetric monoidal combinatorial model category structure, see [11, Section 4, Example 4.12]. The construction roughly consists of two steps. One first defines a certain nice model structure whose weak equivalences are quasi-isomorphisms of complexes of sheaves. In the next step one takes a left Bousfield localization of the model structure at $\mathbb{A}^1$-homotopy. Using a generalization of the construction of symmetric spectra, one can "stabilize" the tensor operation with a shifted Tate object over X in $\mathrm{Comp}(\mathcal{N}^{tr}(X))$, so that one obtains a new category $Sp_{\mathrm{Tate}}(X)$ endowed with a stable symmetric monoidal combinatorial model category (see [11, Proposition 7.13, Example 7.15]).

Let $\phi : Y \to X$ be a morphism of smooth schemes. It gives rise to a Quillen adjunction

$$\phi^* : Sp_{\mathrm{Tate}}(X) \rightleftarrows Sp_{\mathrm{Tate}}(Y) : \phi_*$$

where ϕ^* is a symmetric monoidal left Quillen functor. We further suppose that ϕ is smooth separated of finite type, then there is a Quillen adjunction

$$\phi_\sharp : Sp_{\mathrm{Tate}}(Y) \rightleftarrows Sp_{\mathrm{Tate}}(X) : \phi^*.$$

In this case, ϕ^* is both a left Quillen functor and a right Quillen functor. Thus, it preserves (trivial) fibrations and (trivial) cofibrations. Moreover, by Ken Brown's lemma we see that ϕ^* preserves arbitrary weak equivalences.

We let $\mathsf{DM}^{\otimes}_{eff}(X)$ be the symmetric monoidal stable presentable ∞-category, which is obtained from the full subcategory of cofibrant objects $\mathrm{Comp}(\mathcal{N}^{tr}(X))^c$ by inverting weak equivalences. We refer to it as the symmetric monoidal ∞-category of effective mixed motives over X. Similarly, $\mathsf{DM}^{\otimes}(X)$ is defined to be the symmetric monoidal stable presentable ∞-category obtained from $Sp_{\mathrm{Tate}}(X)^c$ by inverting weak equivalences. We call $\mathsf{DM}^{\otimes}(X)$ the symmetric monoidal stable persentable ∞-category of mixed motives over X. We refer to K as the coefficient ring of $\mathsf{DM}^{\otimes}(X)$. We write $\mathbf{1}_X$ for a unit object of $\mathsf{DM}^{\otimes}(X)$. We write $\mathbf{1}_X(n)$ for the Tate object for $n \in \mathbb{Z}$. Given an object M of $\mathsf{DM}(X)$, we usually write $M(n)$ for the tensor product $M \otimes \mathbf{1}_X(n)$ in $\mathsf{DM}(X)$. The tensor product $\mathsf{DM}(X) \times \mathsf{DM}(X) \to \mathsf{DM}(X)$ on $\mathsf{DM}^{\otimes}(X)$ preserves small colimits separately in each variable. The detail construction can be found in [24, Section 5.1] (the notation is slightly different, and X is assumed to be the Zariski spectrum of a perfect field in [24], but it works for a noetherian regular scheme X). The homotopy category of the full subcategory of $\mathsf{DM}(\operatorname{Spec} k)$ spanned by compact objects can be identified with the triangulated category of geometric motives constructed by Voevodsky [50].

Let $f : X \to \operatorname{Spec} k$ be the structure morphism. Since we have the restriction of the symmetric monoidal left Quillen functor $f^* : Sp_{\mathrm{Tate}}(\operatorname{Spec} k)^c \to Sp_{\mathrm{Tate}}(X)^c$ between full subcategories of cofibrant objects, inverting weak equivalences we have a symmetric monoidal colimit-preserving functor

$$f^* : \mathsf{DM}^{\otimes}(k) := \mathsf{DM}^{\otimes}(\operatorname{Spec} k) \to \mathsf{DM}^{\otimes}(X).$$

By abuse of notation, we use the same notation for the induced functor between ∞-categories. By relative adjoint functor theorem [33, 7.3.2.6, 7.3.2.13], there is the right adjoint functor $f_* : \mathsf{DM}(X) \to \mathsf{DM}(k)$ which is lax symmetric monoidal. It gives rise to an adjunction

$$f^* : \mathrm{CAlg}(\mathsf{DM}^{\otimes}(k)) \rightleftarrows \mathrm{CAlg}(\mathsf{DM}^{\otimes}(X)) : f_*.$$

In particular, f_* carries a commutative algebra object M to a commutative algebra object $f_*(M)$ in $\mathsf{DM}^{\otimes}(k)$. For any smooth scheme X, $\mathrm{CAlg}(\mathsf{DM}^{\otimes}(X))$ is a presentable ∞-category (cf. [33, 3.2.3.5]). There is another left Quillen functor $f_\sharp : Sp_{\mathrm{Tate}}(X) \to Sp_{\mathrm{Tate}}(\operatorname{Spec} k)$. The restriction $Sp_{\mathrm{Tate}}(X)^c \to Sp_{\mathrm{Tate}}(\operatorname{Spec} k)^c$ to cofibrant objects preserves weak equivalences, and therefore inverting weak equivalences induces $f_\sharp : \mathsf{DM}(X) \to \mathsf{DM}(k)$. It determines an adjunction

$$f_\sharp : \mathsf{DM}(X) \rightleftarrows \mathsf{DM}(k) : f^*.$$

We put $M(X) := f_\sharp f^*(\mathbf{1}_k)$ where $\mathbf{1}_k$ is the unit of $\mathsf{DM}(k)$.

Let us consider the unit object $\mathbf{1}_X = f^*(\mathbf{1}_k)$ in $\mathsf{DM}^{\otimes}(X)$ which we regard as a commutative algebra object in $\mathsf{DM}^{\otimes}(X)$. The image $f_*(\mathbf{1}_X) = f_*f^*(\mathbf{1}_k)$ is a commutative algebra object in $\mathsf{DM}^{\otimes}(k)$, namely, $f_*(\mathbf{1}_X)$ in $\mathrm{CAlg}(\mathsf{DM}^{\otimes}(k))$.

Definition 3.1. Let X be a smooth scheme separated of finite type over k. We define an object M_X in $\mathrm{CAlg}(\mathsf{DM}^{\otimes}(k))$ to be $f_*(\mathbf{1}_X)$. We shall refer to M_X as the *cohomological motivic algebra* of X with coefficients in K.

Remark 3.2. This algebra M_X will play a role of a motivic analog of the singular cochain complex $C^*(S, K)$ of a topological space S that is endowed with a structure of an E_∞-algebra. Our principle is that one may consider M_X to be a motivic homotopy type of X with coefficients in K, that occurs in the title of this paper. On the other hand, $M(X)$ is a motivic counterpart of the singular chain complex $C_*(S, K)$.

Remark 3.3. There should be several approaches to a generalization to singular varieties. Although we will treat smooth schemes in this paper, we give a brief outline of an approach based on the theory of Beilinson motives [13]: We will define M_X. Suppose that the coefficient ring K is a field of characteristic zero and $f : X \to \operatorname{Spec} k$ is a (possibly singular) scheme X which is separated of finite type over k. Thanks to [13], we have the symmetric monoidal ∞-category $\mathsf{DM}_B^{\otimes}(X)$ of Beilinson motives. Using the adjunctions

$$f_! : \mathsf{DM}_B(X) \rightleftarrows \mathsf{DM}_B(k) : f^!, \qquad f^* : \mathsf{DM}_B(k) \rightleftarrows \mathsf{DM}_B(X) : f_*$$

for Beilinson motives, we define $M(X) \in \mathsf{DM}_B(k)$ to be $f_!f^!(\mathbf{1}_k)$ and define $M_X \in \mathrm{CAlg}(\mathsf{DM}_B^{\otimes}(k))$ to be $f_*f^*(\mathbf{1}_k)$. We refer the reader to [13, 15.2, 16.1] for the six functors formalism and the comparison between $\mathsf{DM}_B(k)$ and $\mathsf{DM}(k)$. By [13, 4.4.17], $M(X)$ is dualizable. The associated dualizing functor "exchanges $*$ and !" [13, 15.2.4] and $\mathbf{1}_k$ is a dualizing object over the base field k, so that the dual $M(X)^\vee$ is naturally equivalent to M_X in $\mathsf{DM}_B(k) \simeq \mathsf{DM}(k)$ (cf. Proposition 3.4 below).

3.2 We consider functoriality of cohomological motivic algebras. Let $f : X \to \operatorname{Spec} k$ and $g : Y \to \operatorname{Spec} k$ be two smooth scheme separated of finite type over k. Let $\phi : Y \to X$ be a morphism over k. As above, there is an adjunction $\phi^* : \mathrm{CAlg}(\mathsf{DM}^\otimes(X)) \rightleftarrows \mathrm{CAlg}(\mathsf{DM}^\otimes(Y)) : \phi_*$. If we write M_Y for $g_*(\mathbf{1}_Y)$ we have a morphism

$$M_X = f_*(\mathbf{1}_X) \to f_*\phi_*\phi^*(\mathbf{1}_X) \simeq g_*(\mathbf{1}_Y) = M_Y$$

in $\mathrm{CAlg}(\mathsf{DM}^\otimes(k))$ where the first map is induced by the unit map $\mathbf{1}_X \to \phi_*\phi^*(\mathbf{1}_X) \simeq \phi_*(\mathbf{1}_Y)$. Thus, the assignment $X \mapsto M_X$ is contravariantly functorial with respect to X. We will write $\phi^* : M_X \to M_Y$ for this morphism in $\mathrm{CAlg}(\mathsf{DM}^\otimes(k))$ or in the underlying category $\mathsf{DM}(k)$. Unfortunately, the notation ϕ^* in $\phi^* : M_X \to M_Y$ overlaps with $\phi^* : \mathsf{DM}(X) \to \mathsf{DM}(Y)$ or $\phi^* : \mathrm{CAlg}(\mathsf{DM}^\otimes(X)) \to \mathrm{CAlg}(\mathsf{DM}^\otimes(Y))$ though these have different meanings. We hope that it causes no confusion. The assignment $X \mapsto M(X)$ is covariantly functorial. For $\phi : Y \to X$, consider the unit map $u : \mathbf{1}_X \to f^*f_\sharp(\mathbf{1}_X)$. We then have

$$M(Y) = g_\sharp(\mathbf{1}_Y) \simeq g_\sharp\phi^*(\mathbf{1}_X) \xrightarrow{g_\sharp\phi^*(u)} g_\sharp\phi^*f^*f_\sharp(\mathbf{1}_X) \simeq g_\sharp g^* f_\sharp(\mathbf{1}_X) \to f_\sharp(\mathbf{1}_X) = M(X)$$

where the final arrow is induced by the counit $g_\sharp g^* \to \mathrm{id}$. Let Sm_k be the nerve of the category of smooth schemes separated of finite type over k. We will give a functorial construction $X \mapsto M_X$ which is defined as a functor $\Xi : \mathrm{Sm}_k^{op} \to \mathrm{CAlg}(\mathsf{DM}^\otimes(k))$. The result is summarized as follows:

Proposition 3.4. *Let $M(-) : \mathrm{Sm}_k \to \mathsf{DM}(k)$ be the functor which carries X to $M(X)$. We define $\mathsf{Hom}_{\mathsf{DM}(k)}(-, \mathbf{1}_k) : \mathsf{DM}(k)^{op} \to \mathsf{DM}(k)$ to be the functor which carries M to $M^\vee = \mathsf{Hom}_{\mathsf{DM}(k)}(M, \mathbf{1}_k)$. By $\mathsf{Hom}(-,-)$ we indicate the internal Hom object. (We will make a construction of these functors below.) Let $M(-)^\vee : \mathrm{Sm}_k^{op} \to \mathsf{DM}(k)$ be the composite of the above two functors, which carries X to $M(X)^\vee$. Then there is a functor $\Xi : \mathrm{Sm}_k^{op} \to \mathrm{CAlg}(\mathsf{DM}^\otimes(k))$ which makes the diagram commutative*

$$\begin{array}{ccc} & & \mathrm{CAlg}(\mathsf{DM}^\otimes(k)) \\ & \overset{\Xi}{\nearrow} & \downarrow \\ \mathrm{Sm}_k^{op} & \xrightarrow[M(-)^\vee]{} & \mathrm{h}(\mathsf{DM}(k)) \end{array}$$

where the right vertical arrow is the forgetful functor.

We first construct $\Xi : \mathrm{Sm}_k^{op} \to \mathrm{CAlg}(\mathsf{DM}^\otimes(k))$. The busy readers are invited to skip the remainder for the time being and proceed to Section 3.3 or 3.4. We consider the following general situation. The functor Ξ will appear in Example 3.6 as an example of the following setup. Let I be the nerve of a category. Suppose that I has a final object $\star \in I$. We are mainly interested in the case $I = \mathrm{Sm}_k$. Let us consider a family $\{\mathbb{M}(X)\}_{X \in I}$ of symmetric monoidal model categories indexed by I. More precisely, we assign a combinatorial symmetric monoidal model category $\mathbb{M}(X)$ to any $X \in I$ (we here assume that a unit is cofibrant) and assign a symmetric monoidal left Quillen functor $\phi^* : \mathbb{M}(X) \to \mathbb{M}(Y)$ to any morphism $Y \to X$

in I. Moreover, suppose that for $\phi \circ \psi : Z \to Y \to X$ there is a structural natural equivalence $\psi^*\phi^* \simeq (\phi\circ\psi)^*$. Main example is the family $\{Sp_{\mathrm{Tate}}(X)\}_{X\in \mathrm{Sm}_k}$. Consider the pair $(\mathbb{M}^c(X), W^c_X)$ such that $\mathbb{M}^c(X)$ is the full subcategory of cofibrant objects in the model category $\mathbb{M}(X)$, and W^c_X is the collection of weak equivalences in $\mathbb{M}(X)^c$. We think of this pair as the nerve of a category $\mathbb{M}^c(X)$ endowed with the collection of morphisms, determined by W^c_X. We apply to the assignment $X \mapsto (\mathbb{M}(X)^c, W^c_X)$ the construction in [33, Section 4.1.7.1, 4.1.7.2] of inverting weak equivalences in symmetric monoidal categories in the functorial way. We then get a functor

$$d : I^{op} \to \mathrm{CAlg}(\widehat{\mathrm{Cat}}_\infty)$$

which carries X to $\mathbb{M}^{\otimes}_{\infty}(X) := \mathbb{M}^c(X)[(W^c_X)^{-1}]$. Here $\mathbb{M}^c(X)[(W^c_X)^{-1}]$ is the symmetric monoidal ∞-category obtained from $\mathbb{M}(X)^c$ by inverting W^c_X. The symmetric monoidal structure on $\widehat{\mathrm{Cat}}_\infty$ is given by cartesian products, and $\mathrm{CAlg}(\widehat{\mathrm{Cat}}_\infty)$ is naturally identified with the ∞-category of symmetric monoidal (large) ∞-categories whose morphisms are symmetric monoidal functors, cf. [33]. Recall that the ∞-category $\mathrm{CAlg}(\widehat{\mathrm{Cat}}_\infty)$ can be realized as the full subcategory of $\mathrm{Fun}(\Gamma, \widehat{\mathrm{Cat}}_\infty)$ spanned by commutative monoid objects, where Γ is the nerve of the category of pointed finite sets. The functor $d : I^{op} \to \mathrm{CAlg}(\widehat{\mathrm{Cat}}_\infty) \subset \mathrm{Fun}(\Gamma, \widehat{\mathrm{Cat}}_\infty)$ induces a functor $I^{op} \times \Gamma \to \widehat{\mathrm{Cat}}_\infty$. Applying the relative nerve functor to $I^{op} \times \Gamma \to \widehat{\mathrm{Cat}}_\infty$ (cf. [32, 3.2.5]), we have a coCartesian fibration

$$D : \mathcal{E} \to I^{op} \times \Gamma$$

such that each restriction $\mathcal{E}_X := D^{-1}(\{X\} \times \Gamma) \to \{X\} \times \Gamma$ is a symmetric monoidal ∞-category equivalent to $\mathbb{M}^{\otimes}_\infty(X)$. Let $P : \overline{\mathrm{CAlg}}(\mathcal{E}) \to I^{op}$ be a map of simplicial sets defined as follows. For $q : K \to I^{op}$, the set of $K \to \overline{\mathrm{CAlg}}(\mathcal{E})$ over q is defined to be the set of maps $K \times \Gamma \to \mathcal{E}$ extending $q \times \mathrm{id} : K \times \Gamma \to I^{op} \times \Gamma$. Namely, it is $\mathrm{Fun}(\Gamma, \mathcal{E}) \times_{\mathrm{Fun}(\Gamma, I^{op}\times\Gamma)} I^{op} \overset{\mathrm{pr}_2}{\to} I^{op}$ where $I^{op} \to \mathrm{Fun}(\Gamma, I^{op} \times \Gamma)$ is induced by the identity of $I^{op} \times \Gamma$. By the stability property [32, 3.1.2.1 (1), 2.4.2.3. (2)] of coCartesian fibrations, $\overline{\mathrm{CAlg}}(\mathcal{E}) \to I^{op}$ is a coCartesian fibration. Let $\mathrm{CAlg}(\mathcal{E})$ be the largest subcomplex of $\overline{\mathrm{CAlg}}(\mathcal{E})$ that consists of those vertices $v \in \overline{\mathrm{CAlg}}(\mathcal{E})$ such that $\{P(v)\} \times \Gamma \to \mathcal{E}$ determines a commutative algebra object of $\mathcal{E}_{P(v)}$. According to [32, 3.1.2.1 (2)] the induced map $\mathrm{CAlg}(\mathcal{E}) \to I^{op}$ is also a coCartesian fibration. Note that by the construction, for each X in I the fiber over X is $\mathrm{CAlg}(\mathcal{E}_X) \simeq \mathrm{CAlg}(\mathbb{M}^{\otimes}_\infty(X))$, and for each $\phi : Y \to X$ in I the induced map $\mathrm{CAlg}(\mathbb{M}^{\otimes}_\infty(X)) \to \mathrm{CAlg}(\mathbb{M}^{\otimes}_\infty(Y))$ is equivalent to the pullback functor ϕ^*. Each $(\mathbb{M}(X)^c, W^c_X)$ admits a symmetric monoidal functor $(\mathbb{M}^c(\star), W^c_\star) \to (\mathbb{M}(X)^c, W^c_X)$ induced by the morphism $X \to \star$, which preserves weak equivalences. If $d_\star : I^{op} \to \mathrm{CAlg}(\widehat{\mathrm{Cat}}_\infty)$ denotes the constant functor taking value $\mathbb{M}^{\otimes}_\infty(\star)$, it gives rise to a natural transformation $d_\star \to d$. By using the relative nerve functor as above, one has a map between coCartesian fibrations

$$\begin{array}{ccc} I^{op} \times \mathbb{M}^{\otimes}_\infty(\star) & \xrightarrow{F^*_\circ} & \mathcal{E} \\ & {\scriptstyle \mathrm{id}\times e} \searrow \quad \swarrow {\scriptstyle D} & \\ & I^{op} \times \Gamma & \end{array}$$

where $e : \mathbb{M}^{\otimes}_\infty(\star) \to \Gamma$ is a coCartesian fibration that determines the symmetric monoidal ∞-category $\mathbb{M}^{\otimes}_\infty(\star)$. The horizontal map preserves coCartesian edges. Apply the same construction of $\mathrm{CAlg}(\mathcal{E}) \to I^{op}$ to $I^{op} \times \mathbb{M}^{\otimes}_\infty(\star) \to I^{op} \times \Gamma$, we obtain the constant coCartesian fibration $I^{op} \times \mathrm{CAlg}(\mathbb{M}^{\otimes}_\infty(\star)) \to I^{op}$ and a map of coCartesian fibrations

$$F^* : I^{op} \times \mathrm{CAlg}(\mathbb{M}^{\otimes}_\infty(\star)) \to \mathrm{CAlg}(\mathcal{E})$$

over I^{op}. For each $f : X \to \star$ in I, the fiber $\mathrm{CAlg}(\mathbb{M}^{\otimes}_{\infty}(\star)) \to \mathrm{CAlg}(\mathcal{E}_X) \simeq \mathrm{CAlg}(\mathbb{M}^{\otimes}_{\infty}(X))$ over X is equivalent to f^*. Thus, each fiber admits the right adjoint functor $f_* : \mathrm{CAlg}(\mathbb{M}^{\otimes}_{\infty}(X)) \to \mathrm{CAlg}(\mathbb{M}^{\otimes}_{\infty}(\star))$. In addition, F^* preserves coCartesian edges. Therefore by the relative adjoint functor theorem [33, 7.3.2.6] there is a relative right adjoint $F_* : \mathrm{CAlg}(\mathcal{E}) \to I^{op} \times \mathrm{CAlg}(\mathbb{M}^{\otimes}_{\infty}(\star))$ over I^{op}. (We refer to [33, 7.3.2] for the notion of relative adjoint functor.) For each $f : X \to \star$, the fiber $\mathrm{CAlg}(\mathbb{M}^{\otimes}_{\infty}(X)) \to \mathrm{CAlg}(\mathbb{M}^{\otimes}_{\infty}(\star))$ is equivalent to f_*.

Now we define a functorial assignment $X \mapsto f_*(\mathbf{1}_{\mathbb{M}(X)})$ where $\mathbf{1}_{\mathbb{M}(X)}$ is a unit of $\mathbb{M}(X)$ and f is the natural morphism $X \to \star$. We let $\iota : I^{op} \to \mathrm{CAlg}(\mathbb{M}^{\otimes}_{\infty}(\star))$ be the constant functor whose value is the unit $\mathbf{1}_\star$ of $\mathbb{M}^{\otimes}_{\infty}(\star)$. It yields a section $\mathrm{id} \times \iota : I^{op} \to I^{op} \times \mathrm{CAlg}(\mathbb{M}^{\otimes}_{\infty}(\star))$. Composing it with F^*, we obtain a section $S : I^{op} \to \mathrm{CAlg}(\mathcal{E})$ of $\mathrm{CAlg}(\mathcal{E}) \to I^{op}$ which carries X to a unit in $\mathrm{CAlg}(\mathcal{E}_X) \simeq \mathrm{CAlg}(\mathbb{M}^{\otimes}_{\infty}(X))$ (every edge in I^{op} maps to a canonical coCartesian edge). We define $I^{op} \to \mathrm{CAlg}(\mathbb{M}^{\otimes}_{\infty}(\star))$ to be the composite

$$\Phi : I^{op} \stackrel{S}{\to} \mathrm{CAlg}(\mathcal{E}) \stackrel{F_*}{\to} I^{op} \times \mathrm{CAlg}(\mathbb{M}^{\otimes}_{\infty}(\star)) \stackrel{\mathrm{pr}_2}{\to} \mathrm{CAlg}(\mathbb{M}^{\otimes}_{\infty}(\star)).$$

Remark 3.5. We give a little bit more conceptual explanation of Φ. Let $\mathcal{C} \to \mathcal{O}$ and $\mathcal{D} \to \mathcal{O}$ be categorical fibrations over an ∞-category $\mathcal{O}$. Let $\alpha : \mathcal{C} \rightleftarrows \mathcal{D} : \beta$ be functors over $\mathcal{O}$. Suppose that α is a left adjoint to β. Observe that compositions with α and β induce an adjoint pair between functor categories $\mathrm{Fun}(\mathcal{O},\mathcal{C}) \rightleftarrows \mathrm{Fun}(\mathcal{O},\mathcal{D})$. To see this, if $\mathcal{M} \to \Delta^1$ is both a coCartesian fibration and a Cartesian fibration which represents the adjoint pair (α,β) (cf. [32, 5.5.2.1]), the projection $\mathrm{Fun}(\mathcal{O},\mathcal{M}) \times_{\mathrm{Fun}(\mathcal{O},\Delta^1)} \Delta^1 \to \Delta^1$ is both a coCartesian fibration and a Cartesin fibration that induces an adjoint pair between functor categories, where $\Delta^1 \to \mathrm{Fun}(\mathcal{O},\Delta^1)$ is determined by the projection $\mathcal{O} \times \Delta^1 \to \Delta^1$. Suppose further that α is a left adjoint to β relative to $\mathcal{O}$ (cf. [33, 7.3.2.2]). The restriction of the above adjunction induces

$$\mathrm{Sect}(\alpha) : \mathrm{Sect}_{\mathcal{O}}(\mathcal{C}) := \mathrm{Fun}_{\mathcal{O}}(\mathcal{O},\mathcal{C}) \rightleftarrows \mathrm{Fun}_{\mathcal{O}}(\mathcal{O},\mathcal{D}) = \mathrm{Sect}_{\mathcal{O}}(\mathcal{D}) : \mathrm{Sect}(\beta).$$

We deduce from [33, 7.3.2.5] that this pair is an adjunction. We now apply this to

$$F^* : I^{op} \times \mathrm{CAlg}(\mathbb{M}^{\otimes}_{\infty}(\star)) \rightleftarrows \mathrm{CAlg}(\mathcal{E}) : F_*$$

over I^{op}. We then have the induced adjunction

$$\mathrm{Sect}(F^*) : \mathrm{Fun}(I^{op}, \mathrm{CAlg}(\mathbb{M}^{\otimes}_{\infty}(\star))) \simeq \mathrm{Sect}_{I^{op}}(I^{op} \times \mathrm{CAlg}(\mathbb{M}^{\otimes}_{\infty}(\star))) \rightleftarrows \mathrm{Sect}_{I^{op}}(\mathrm{CAlg}(\mathcal{E})) : \mathrm{Sect}(F_*).$$

If $\iota \in \mathrm{Fun}(I^{op}, \mathrm{CAlg}(\mathbb{M}^{\otimes}_{\infty}(\star)))$ is the constant functor with value $\mathbf{1}_\star$, the unit transformation $\mathrm{id} \to \mathrm{Sect}(F_*) \circ \mathrm{Sect}(F^*)$ induces $\iota \to \mathrm{Sect}(F_*) \circ \mathrm{Sect}(F^*)(\iota) = \Phi$.

Example 3.6. Let $I = \mathrm{Sm}_k$ and $\star = \mathrm{Spec}\, k$. Let $\mathbb{M}(X) = Sp_{\mathrm{Tate}}(X)$. We define

$$\Xi : \mathrm{Sm}_k^{op} \to \mathrm{CAlg}(\mathsf{DM}^{\otimes}(k))$$

to be Φ. Unfolding our construction we see that Ξ carries X to M_X, and $\phi : Y \to X$ maps to $\phi^* : M_X \to M_Y$.

Remark 3.7. Let $I = \mathrm{Sm}_k$ and $\star = \mathrm{Spec}\, k$. Let $\mathbb{M}(X) = \mathrm{Comp}(\mathcal{N}^{tr}(X))$. In this case, the above construction also works. But we will not consider this setting: $f_*(\mathbf{1}_X)$ is not an appropriate object we want to consider (for example, Theorem 4.3 does not hold). It is important to adopt $\mathsf{DM}^{\otimes}(k)$ instead of $\mathsf{DM}^{\otimes}_{eff}(k)$ in Definition 3.1 and Propistion 3.4.

Example 3.8. Let I be the category Sch of separated and quasi-compact schemes. For any X in Sch, we let $\mathrm{Comp}(X)$ be the symmetric monoidal category of (possibly unbounded) cochain complexes of quasi-coherent sheaves on X. According to [11, Example 2.3, 3.1, 3.2], there is a symmetric monoidal model structure on $\mathrm{Comp}(X)$ such that weak equivalences are quasi-isomorphisms, and for any $Y \to X$ in Sch the pullback functor $\mathrm{Comp}(X) \to \mathrm{Comp}(Y)$ is a left Quillen functor. Put $\mathrm{Comp}(X) = \mathbb{M}(X)$. One can apply to this setting our construction and obtain $\mathrm{Sch}^{op} \to \mathrm{CAlg}(\mathbb{M}^{\otimes}_{\infty}(\mathrm{Spec}\,\mathbb{Z}))$.

Next we define a functor $\mathrm{Sm}_k \to \mathsf{DM}(k)$ which carries X to $M(X)$. In some sense, the construction is the dual of that of Ξ and is easier. We continue to work with the family $\{\mathbb{M}(X)\}$. Assume that for each $f : X \to \star$ in I, $f^* : \mathbb{M}(\star) \to \mathbb{M}(X)$ is also right Quillen functor (therefore, it preserves arbitrary weak equivalences). We denote by $f_\sharp : \mathbb{M}(X) \to \mathbb{M}(\star)$ the left adjoint. Applying the "dual version" of the relative nerve functor or the unstraightning functor to $X \mapsto \mathbb{M}_\infty(X)$, we obtain a Cartesian fibration $\mathcal{F} \to I$. For each $X \in I$, its fiber is equivalent to $\mathbb{M}_\infty(X)$. Notice that it is not a coCartesian fibration but a Cartesian fibration. As in the case of $\mathcal{E} \to I^{op}$, the natural pullback functors $\mathbb{M}(\star) \to \mathbb{M}(X)$ induce a morphism of Cartesian fibrations

$$\begin{array}{ccc} \mathcal{F} & \xleftarrow{G^*} & I \times \mathbb{M}_\infty(\star) \\ & \searrow \quad \swarrow & \\ & I. & \end{array}$$

where $I \times \mathbb{M}_\infty(\star) \to I$ is the projection that is regarded as a Cartesian fibration corresponding to the constant functor $I \to \widehat{\mathrm{Cat}}_\infty$ with value $\mathbb{M}_\infty(\star)$. Each fiber of the horizontal map over $X \in I$ is equivalent to f^* where $f : X \to \star$ is the natural morphism. Therefore it admits a left adjoint functor $f_\sharp : \mathbb{M}_\infty(X) \to \mathbb{M}_\infty(\star)$. Moreover, G^* preserves Cartesian edges. Thus, by the relative adjoint functor theorem [33, 7.3.2.6] there is a left adjoint $G_\sharp : \mathcal{F} \to I \times \mathbb{M}_\infty(\star)$ relative to I. (Its fiber over $X \in I$ is equivalent to $f_\sharp$.) Let $u : I \to I \times \mathbb{M}_\infty(\star)$ be the functor determined by the identity $I \to I$ and the constant functor $I \to \mathbb{M}_\infty(\star)$ taking the value $\mathbf{1}_\star$. Then $\Psi : I \to \mathbb{M}_\infty(\star)$ is defined to be the composite

$$I \xrightarrow{u} I \times \mathbb{M}_\infty(\star) \xrightarrow{G^*} \mathcal{F} \xrightarrow{G_\sharp} I \times \mathbb{M}_\infty(\star) \xrightarrow{\mathrm{pr}_2} \mathbb{M}_\infty(\star).$$

Example 3.9. Let $I = \mathrm{Sm}_k$ and $\star = \mathrm{Spec}\,k$. Let $\mathbb{M}(X) = Sp_{\mathrm{Tate}}(X)$. We define $M(-) : \mathrm{Sm}_k \to \mathsf{DM}(k)$ to be Ψ. By our construction, it sends X to an object equivalent to $M(X)$.

We define a functor $\mathsf{Hom}_{\mathsf{DM}(X)}(-, \mathbf{1}_X) : \mathsf{DM}(X)^{op} \to \mathsf{DM}(X)$ as follows. We let

$$\mathsf{Hom}_{Sp_{\mathrm{Tate}}(X)}(-, \mathbf{1}'_X) : (Sp_{\mathrm{Tate}}(X)^c)^{op} \to Sp_{\mathrm{Tate}}(X)$$

be the functor given by $M \mapsto \mathsf{Hom}_{Sp_{\mathrm{Tate}}(X)}(M, \mathbf{1}'_X)$, where $\mathsf{Hom}_{Sp_{\mathrm{Tate}}(X)}(-,-)$ denotes the internal Hom object in $Sp_{\mathrm{Tate}}(X)$, and $\mathbf{1}'_X$ is a fibrant model of the unit $\mathbf{1}_X$. By the axiom of symmetric monoidal model category, the functor $\mathsf{Hom}_{Sp_{\mathrm{Tate}}(X)}(-, \mathbf{1}'_X)$ preserves weak equivalences. We define $\mathsf{Hom}_{\mathsf{DM}(X)}(-, \mathbf{1}_X) : \mathsf{DM}(X)^{op} \to \mathsf{DM}(X)$ to be the functor obtained from $\mathsf{Hom}_{Sp_{\mathrm{Tate}}(X)}(-, \mathbf{1}'_X)$ by inverting weak equivalences.

Proof of Proposition 3.4. We have constructed the functor $\Xi : \mathrm{Sm}_k^{op} \to \mathrm{CAlg}(\mathsf{DM}^{\otimes}(k))$ and $M(-) : \mathrm{Sm}_k \to \mathsf{DM}(k)$ in Example 3.6 and 3.9. For simplicity, we write Ξ also for the composite $\mathrm{Sm}_k^{op} \xrightarrow{\Xi} \mathrm{CAlg}(\mathsf{DM}^{\otimes}(k)) \to \mathsf{DM}(k)$. We first observe that for $f : X \to \mathrm{Spec}\,k$ in Sm_k there is a canonical equivalence $M(X)^\vee \xrightarrow{\sim} M_X = f_*(\mathbf{1}_X)$. Actually, this equivalence follows from the equivalences of mapping spaces

$$\begin{aligned}
\mathrm{Map}_{\mathsf{DM}(k)}(M, \mathsf{Hom}_{\mathsf{DM}(k)}(f_\sharp \mathbf{1}_X, \mathbf{1}_k)) &\simeq \mathrm{Map}_{\mathsf{DM}(k)}(M \otimes f_\sharp \mathbf{1}_X, \mathbf{1}_k) \\
&\simeq \mathrm{Map}_{\mathsf{DM}(k)}(f_\sharp(\mathbf{1}_X), \mathsf{Hom}_{\mathsf{DM}(k)}(M, \mathbf{1}_k)) \\
&\simeq \mathrm{Map}_{\mathsf{DM}(X)}(\mathbf{1}_X, f^*\mathsf{Hom}_{\mathsf{DM}(k)}(M, \mathbf{1}_k)) \\
&\simeq \mathrm{Map}_{\mathsf{DM}(X)}(\mathbf{1}_X, \mathsf{Hom}_{\mathsf{DM}(X)}(f^*(M), f^*(\mathbf{1}_k))) \\
&\simeq \mathrm{Map}_{\mathsf{DM}(X)}(f^*(M), \mathbf{1}_X) \\
&\simeq \mathrm{Map}_{\mathsf{DM}(k)}(M, f_*(\mathbf{1}_X))
\end{aligned}$$

for any $M \in \mathsf{DM}(k)$. The equvalences follows from adjunctions $(f_\sharp, f^*)$, (f^*, f_*) and the equivalence $f^*\mathsf{Hom}_{\mathsf{DM}(k)}(M, \mathbf{1}_k) \simeq \mathsf{Hom}_{\mathsf{DM}(X)}(f^*(M), f^*(\mathbf{1}_k))$. If we take $M = \mathsf{Hom}_{\mathsf{DM}(k)}(f_\sharp \mathbf{1}_X, \mathbf{1}_k) = M(X)^\vee$, then the identity of M corresponds to $M(X)^\vee \xrightarrow{\sim} f_*(\mathbf{1}_X)$. The equivalence $M(X)^\vee = f_\sharp(\mathbf{1}_X)^\vee \to f_*(\mathbf{1}_X)$ comes from the dual of $\mathbf{1}_X \to f^* f_\sharp(\mathbf{1}_X)$:

$$f^*(f_\sharp(\mathbf{1}_X)^\vee) \simeq (f^* f_\sharp(\mathbf{1}_X))^\vee \to \mathbf{1}_X$$

and the composition with $f_\sharp(\mathbf{1}_X)^\vee \to f_* f^*(f_\sharp(\mathbf{1}_X)^\vee)$ where $(-)^\vee$ denotes the weak dual, that is, $\mathsf{Hom}_{\mathsf{DM}(-)}(-, \mathbf{1}_{(-)})$. By the functoriality of adjoint maps, it is easy to check that $M(X)^\vee = f_\sharp(\mathbf{1}_X)^\vee \to f_*(\mathbf{1}_X)$ is functorial with respect to $X \in \mathrm{Sm}_k$ at the level of homotopy category, namely, the functor $\mathrm{Sm}_k^{op} \stackrel{M(-)^\vee}{\to} \mathsf{DM}(k) \to \mathrm{h}(\mathsf{DM}(k))$ is naturally equivalent to $\mathrm{Sm}_k^{op} \stackrel{\Xi}{\to} \mathsf{DM}(k) \to \mathrm{h}(\mathsf{DM}(k))$. □

3.3 We give some remarks about properties of cohomological motivic algebras.

Remark 3.10. Since M_X is the weak dual $\mathrm{Hom}_{\mathsf{DM}(k)}(M(X), \mathbf{1}_k)$ of $M(X)$, one can observe that $X \mapsto M_X$ satisfies $\mathbb{A}^1$-homotopy invariance and Nisnevich descent property. Namely, for any projection $X \times \mathbb{A}^1 \to X$ with fiber of the affine line $\mathbb{A}^1 = \operatorname{Spec} k[x]$, $M_X \to M_{X\times\mathbb{A}^1}$ is an equivalence in $\mathrm{CAlg}(\mathsf{DM}^\otimes(k))$. For any pullback diagram

$$\begin{array}{ccc}
V \simeq U \times_X Y & \longrightarrow & Y \\
\downarrow & & \downarrow f \\
U & \xrightarrow{j} & X
\end{array}$$

in Sm_k such that f is étale, j is an open immersion and $(Y\backslash V)_{red} \to (X\backslash U)_{red}$ is an isomorphism, the induced morphism $M_X \to M_U \times_{M_V} M_Y$ is an equivalence in $\mathrm{CAlg}(\mathsf{DM}^\otimes(k))$.

The following is the Kunneth formula for cohomological motivic algebras.

Proposition 3.11. *There exist a canonical equivalence $\mathbf{1}_k \simeq \Xi(\operatorname{Spec} k)$. Suppose that X and Y are projective and smooth over* $\operatorname{Spec} k$. *Then there exists a canonical equivalence $\Xi(X) \otimes \Xi(Y) = M_X \otimes M_Y \simeq M_{X\times Y} = \Xi(X \times Y)$.*

Proof. The first assertion is obvious. Next we prove the second assertion. Consider the Cartesian diagram

$$\begin{array}{ccc}
X \times Y & \xrightarrow{q} & Y \\
p\downarrow & & \downarrow g \\
X & \xrightarrow[f]{} & \operatorname{Spec} k.
\end{array}$$

We will prove that $p^* \otimes q^* : M_X \otimes M_Y \to M_{X\times Y}$ induced by $p^* : M_X \to M_{X\times Y}$ and $q^* : M_Y \to M_{X\times Y}$ is an equivalence. For this purpose, we apply the projection formula for the smooth proper morphism f and the base change theorem for smooth proper morphism g [13, Theorem 1]: we have the sequence of morphisms induced by unit maps and counit maps of adjunctions

$$\begin{array}{rcl} f_*(\mathbf{1}_X) \otimes g_*(\mathbf{1}_Y) & \to & f_*f^*(f_*(\mathbf{1}_X) \otimes g_*(\mathbf{1}_Y)) \simeq f_*(f^*f_*(\mathbf{1}_X) \otimes f^*g_*g^*(\mathbf{1}_k)) \\ & \to & f_*(\mathbf{1}_X \otimes f^*g_*g^*(\mathbf{1}_k)) \simeq f_*(f^*g_*g^*(\mathbf{1}_k)) \\ & \to & f_*(p_*p^*f^*g_*g^*(\mathbf{1}_k)) \simeq f_*(p_*q^*g^*g_*g^*(\mathbf{1}_k)) \\ & \to & f_*(p_*q^*g^*(\mathbf{1}_k)) \simeq f_*p_*(\mathbf{1}_{X\times Y}) \end{array}$$

whose composite is an equivalence since the projection formula and the base change theorem imply that the above sequence induces $f_*(\mathbf{1}_X) \otimes g_*(\mathbf{1}_X) \simeq f_*(\mathbf{1}_X \otimes f^*g_*(\mathbf{1}_Y))$ and $f^*g_*(\mathbf{1}_Y) \simeq p_*q^*(\mathbf{1}_Y)$. It will suffice to check that this composite coincides with $p^* \otimes q^*$. It is straightforward to verify that

$$f^*(\mathbf{1}_k) \to f^*g_*g^*(\mathbf{1}_k) \to p_*p^*f^*g_*g^*(\mathbf{1}_k) = p_*q^*g^*g_*g^*(\mathbf{1}_k) \to p_*q^*g^*(\mathbf{1}_k) = p_*p^*f^*(\mathbf{1}_k)$$

is equivalent to $f^*(\mathbf{1}_k) \to p_*p^*f^*(\mathbf{1}_k)$ induced by the unit map $\mathrm{id} \to p_*p^*$. Then we see that $f_*(\mathbf{1}_X) \otimes \mathbf{1}_k \to f_*(\mathbf{1}_X) \otimes g_*(\mathbf{1}_X) \xrightarrow{\sim} f_*p_*p^*(\mathbf{1}_X)$ is equivalent to $p^* : M_X = f_*(\mathbf{1}_X) \to f_*p_*p^*(\mathbf{1}_X) = M_{X\times Y}$. Similarly, $\mathbf{1}_k \otimes g_*(\mathbf{1}_Y) \to f_*(\mathbf{1}_X) \otimes g_*(\mathbf{1}_X) \xrightarrow{\sim} f_*p_*p^*(\mathbf{1}_X)$ is equivalent to $q^* : M_Y \to M_{X\times Y}$. Thus, $p^* \otimes q^* : M_X \otimes M_Y \to M_{X\times Y}$ is an equivalence. □

3.4 We will study various objects in $\mathrm{CAlg}(\mathsf{DM}^{\otimes}(k))$ other than M_X:

Example 3.12. Let $X \in \mathrm{Sm}_k$. Let $x : Y = \operatorname{Spec} k \to X$ and $y : Z = \operatorname{Spec} k \to X$ be two k-rational points on X. Then we have the pushout diagram

$$\begin{array}{ccc} M_X & \xrightarrow{x^*} & M_{\operatorname{Spec} k} \\ {\scriptstyle y^*}\downarrow & & \downarrow \\ M_{\operatorname{Spec} k} & \longrightarrow & M_{\operatorname{Spec} k} \otimes_{M_X} M_{\operatorname{Spec} k}. \end{array}$$

in $\mathrm{CAlg}(\mathsf{DM}^{\otimes}(k))$. Keep in mind that pushouts in $\mathrm{CAlg}(\mathsf{DM}^{\otimes}(k))$ *do not* commute with pushouts in $\mathsf{DM}(k)$ through the forgetful functor. By Proposition 3.11, $M_{\operatorname{Spec} k} \simeq \mathbf{1}_k$. Thus, $M_{\operatorname{Spec} k} \otimes_{M_X} M_{\operatorname{Spec} k} \simeq \mathbf{1}_k \otimes_{M_X} \mathbf{1}_k$ in $\mathrm{CAlg}(\mathsf{DM}^{\otimes}(k))$. We call $P_X(x,y) := \mathbf{1}_k \otimes_{M_X} \mathbf{1}_k$ the *motivic algebra of path torsors* from x to y.

Example 3.13. Consider the object $M_X \otimes_{M_X \otimes M_X} M_X$. Note that $\mathrm{CAlg}(\mathsf{DM}^{\otimes}(k))$ is presentable, and thus $\mathrm{CAlg}(\mathsf{DM}^{\otimes}(k))$ is tensored over $\mathcal{S}$. There is a canonical equivalence $S^1 \otimes M_X \simeq M_X \otimes_{M_X \otimes M_X} M_X$ where S^1 is the circle which belongs to $\mathcal{S}$. Thus, by the functoriality of the tensor operation, $\mathrm{Map}_{\mathcal{S}}(S^1, S^1) \simeq S^1$ naturally acts on $S^1 \otimes M_X$ (it is a version of Connes operator [31]; the precise formulation is left to the reader). We refer to $HHM_X := M_X \otimes_{M_X \otimes M_X} M_X$ as the *motivic algebra of free loop space* of X.

3.5 In Example 3.12, if one supposes $x = y$, then $P_X(x,y)$ has an additional structure. The augmentation $M_X \to \mathbf{1}_k \simeq M_{\mathrm{Spec}\,k}$, induced by $x : \mathrm{Spec}\,k \to X$, gives rise to

$$\mathbf{1}_k \otimes_{M_X} \mathbf{1}_k \simeq \mathbf{1}_k \otimes_{M_X} M_X \otimes_{M_X} \mathbf{1}_k \to \mathbf{1}_k \otimes_{M_X} \mathbf{1}_k \otimes_{M_X} \mathbf{1}_k \simeq (\mathbf{1}_k \otimes_{M_X} \mathbf{1}_k) \otimes (\mathbf{1}_k \otimes_{M_X} \mathbf{1}_k)$$

and $\mathbf{1}_k \otimes_{M_X} \mathbf{1}_k \to \mathbf{1}_k \otimes_{\mathbf{1}_k} \mathbf{1}_k \simeq \mathbf{1}_k$ in $\mathrm{CAlg}(\mathsf{DM}^{\otimes}(k))$. There is also the flip $\mathbf{1}_k \otimes_{M_X} \mathbf{1}_k \simeq \mathbf{1}_k \otimes_{M_X} \mathbf{1}_k$. Informally, these data define a structure of a cogroup object on $\mathbf{1}_k \otimes_{M_X} \mathbf{1}_k$ in $\mathrm{CAlg}(\mathsf{DM}^{\otimes}(k))$. Here $\mathrm{CAlg}(\mathsf{DM}^{\otimes}(k))$ is endowed with the coCartesian monoidal structure given by coproducts. The precise formulation of this structure is as follows. Let Δ_+ be the category of (possibly nonempty) linearly ordered finite sets. Objects are the empty set $[-1]$, $[0] = \{0\}$, $[1] = \{0,1\}$, $[2] = \{0,1,2\}, \ldots$. Note that Δ_+ without $[-1]$ is Δ. Suppose that the morphism $M_X \to \mathbf{1}_k$ is given by a map $\mathrm{N}(\Delta_+^{\le 0}) = \mathrm{N}(\{[-1] \to [0]\}) \to \mathrm{CAlg}(\mathsf{DM}^{\otimes}(k))$. Since $\mathrm{CAlg}(\mathsf{DM}^{\otimes}(k))$ has small colimits (in fact, presentable), the map $\mathrm{N}(\Delta_+^{\le 0}) \to \mathrm{CAlg}(\mathsf{DM}^{\otimes}(k))$ admits a left Kan extension $e : \mathrm{N}(\Delta_+) \to \mathrm{CAlg}(\mathsf{DM}^{\otimes}(k))$. Namely,

$$\mathcal{G}(X,x) = e^{op} : \mathrm{N}(\Delta_+)^{op} \to \mathrm{CAlg}(\mathsf{DM}^{\otimes}(k))^{op}$$

is the Cech nerve associated to $\mathrm{N}(\{[-1] \to [0]\})^{op} \to \mathrm{CAlg}(\mathsf{DM}^{\otimes}(k))^{op}$ (cf. [32, 6.1.2.11]). The evaluation of $\mathcal{G}(X,x)$ at [1] is equivalent to $\mathbf{1}_k \otimes_{M_X} \mathbf{1}_k$. The restriction $\mathrm{N}(\Delta)^{op} \to \mathrm{CAlg}(\mathsf{DM}^{\otimes}(k))^{op}$ is a group object of $\mathrm{CAlg}(\mathsf{DM}^{\otimes}(k))^{op}$ (i.e., a cogroup object in $\mathrm{CAlg}(\mathsf{DM}^{\otimes}(k))$). Namely, it determines a group structure on $\mathbf{1}_k \otimes_{M_X} \mathbf{1}_k$ in $\mathrm{CAlg}(\mathsf{DM}^{\otimes}(k))^{op}$. We refer to e.g. [32, 7.2.2.1] for the notion of group objects.

Next we define an iterated generalization of $\mathcal{G}(X,x)$. Consider the composite $\mathrm{N}(\{[1] \to [0]\})^{op} \subset \mathrm{N}(\Delta_+)^{op} \to \mathrm{CAlg}(\mathsf{DM}^{\otimes}(k))^{op}$ of Cech nerve $\mathcal{G}^{(1)}(X,x) = \mathcal{G}(X,x) : \mathrm{N}(\Delta_+)^{op} \to \mathrm{CAlg}(\mathsf{DM}^{\otimes}(k))^{op}$. There is a unique isomorphism $\mathrm{N}(\Delta_+^{\le 0}) \simeq \mathrm{N}(\{[1] \to [0]\})$. Consider the composite

$$\mathrm{N}(\Delta_+^{\le 0})^{op} \simeq \mathrm{N}(\{[1] \to [0]\})^{op} \subset \mathrm{N}(\Delta_+)^{op} \to \mathrm{CAlg}(\mathsf{DM}^{\otimes}(k))^{op}.$$

Once again, take a rigth Kan extension $\mathcal{G}^{(2)}(X,x) : \mathrm{N}(\Delta_+)^{op} \to \mathrm{CAlg}(\mathsf{DM}^{\otimes}(k))^{op}$ of this composite. Repeating this process n times, we obtain

$$\mathcal{G}^{(n+1)}(X,x) : \mathrm{N}(\Delta_+)^{op} \to \mathrm{CAlg}(\mathsf{DM}^{\otimes}(k))^{op}.$$

By abuse of notation, we write $\mathcal{G}^{(n+1)}(X,x)$ for the group object defined as the restriction $\mathrm{N}(\Delta)^{op} \subset \mathrm{N}(\Delta_+)^{op} \to \mathrm{CAlg}(\mathsf{DM}^{\otimes}(k))^{op}$. (Moreover, one can endow $\mathcal{G}^{(n+1)}(X,x)$ with a structure of an E_{n+1}-monoid, but we will not use this enhanced structure.)

4. Realized motivic rational homotopy type

We will consider the realizations of M_X. The coefficient field K is a field of characteristic zero.

4.1 There are several mixed Weil cohomology theories: singular (Betti) cohomology, l-adic or p-adic étale cohomology, analytic de Rham cohomology, algebraic de Rham cohomology, rigid cohomology, etc (see [13, 17.2] for mixed Weil cohomology). To a mixed Weil cohomology theory E with coefficient field K, one can associate a symmetric monoidal colimit-preserving functor

$$\mathsf{R}_E : \mathsf{DM}^{\otimes}(k) \longrightarrow \mathsf{D}^{\otimes}(K)$$

(see [24, Section 5] for details of the construction in the ∞-categorical setting) which is called the *realization functor* associated to E. Here $\mathsf{D}^{\otimes}(K)$ is the derived ∞-category of K-vector spaces (see Section 2). By the relative adjoint functor theorem [33, 7.3.2.6, 7.3.2.13], the realization functor R_E induces an adjunction

$$\mathrm{CAlg}(\mathsf{R}_E) : \mathrm{CAlg}(\mathsf{DM}^{\otimes}(k)) \rightleftarrows \mathrm{CAlg}(\mathsf{D}^{\otimes}(K)) \simeq \mathrm{CAlg}_K : \mathsf{M}_E$$

where $\mathrm{CAlg}(\mathsf{R}_E)$ is the functor induced by R_E, and M_E is a right adjoint. We shall refer to $\mathrm{CAlg}(\mathsf{R}_E) : \mathrm{CAlg}(\mathsf{DM}^{\otimes}(k)) \to \mathrm{CAlg}_K$ as the *multiplicative realization functor.*

In this section, we consider the realization functor associated to singular cohomology theory:

$$\mathsf{R} : \mathsf{DM}^{\otimes}(k) \longrightarrow \mathsf{D}^{\otimes}(\mathbb{Q}).$$

We here suppose that the base field k is embedded into the complex number field $\mathbb{C}$, and the coefficient field K is $\mathbb{Q}$. This functor sends the object $M(X)$ to a complex $\mathsf{R}(M(X))$ that is quasi-isomorphic to the singular chain complex $C_*(X^t, \mathbb{Q})$ with rational coefficients. Here X^t stands for the underlying topological space of the complex manifold $X \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, \mathbb{C}$. For ease of notation, when no confusion is likely to arise, we often write R for the multiplicative realization functor $\mathrm{CAlg}(\mathsf{R}) : \mathrm{CAlg}(\mathsf{DM}^{\otimes}(k)) \to \mathrm{CAlg}_{\mathbb{Q}}$. Our results in this section relate M_X and its variants to rational homotopy theory; see Theorem 4.3, Remark 4.4, Corollary 4.5, Proposition 4.6.

4.2 There are several algebraic models that describe rational homotopy types of topological spaces. Quillen [43] uses *differential graded (dg) Lie algebras* whereas Sullivan [48] adopts *commutative differential graded (dg) algebras* as models. These approaches are related via Koszul duality between dg Lie algebras and (augmented) commutative dg algebras. In this paper, we use cochain algebras of *polynomial differential forms* introduced by Sullivan as algebraic models of the rational homotopy types of topological spaces.

Let us recall the definition of a cochain algebra of polynomial differential forms on a topological space S, see [16, Section 10] for the comprehensive reference. For a simplicial set P, we let $A_{PL}(P)$ be the commutative differential graded (dg) algebra with rational coefficients of polynomial differential forms. This commutative dg algebra is defined as follows (but we will not need this explicit definition). For each $n \geq 0$, we let Ω_n be the commutative dg algebra of "polynomial differential forms on the standard n-simplex", that is,

$$\Omega_n := \mathbb{Q}[u_0, \ldots, u_n, du_0, \ldots, du_n]/(\Sigma_{i=0}^n u_i - 1, \Sigma_{i=0}^n du_i)$$

where $\mathbb{Q}[u_0, \ldots u_n, du_0, \ldots, du_n]$ is the free commutative graded algebra generated by $u_0, \ldots, u_n$ and $du_0, \ldots, du_n$ with cohomological degrees $|u_i| = 0$, $|du_i| = 1$ for each i, and the differential carries u_i and du_i to du_i and 0, respectively. For any map $f : \Delta^n \to \Delta^m$, the pullback morphism $f^* : \Omega_m \to \Omega_n$ of commutative dg algebras is defined in a natural way (see e.g. [16, Section 10 (c)]). An element of $A_{PL}(P)$ of (cohomological) degree r is data that consists of a collection $\{w_\alpha\}$ indexed by the set of all morphisms $\alpha : \Delta^n \to P$ from standard simplices such that

- each ω_α is an element of Ω_n of degree r,
- $f^*(w_\beta) = w_\alpha$ for any $\alpha : \Delta^n \to P$, $\beta : \Delta^m \to P$, and $f : \Delta^n \to \Delta^m$ such that $\beta \circ f = \alpha$.

The multiplication is given by $\{w_\alpha\} \cdot \{w'_\alpha\} = \{w_\alpha w'_\alpha\}$, and the differential is given by $d\{w_\alpha\} = \{dw_\alpha\}$. If $\phi : P \to P'$ is a map of simplicial sets and $\{\omega_\alpha\}_{\alpha:\Delta^n \to P'}$ is an element of $A_{PL}(P')$, then $\phi^*\{\omega_\alpha\}$ is defined to be $\{\omega_{\phi\circ\beta}\}_{\beta:\Delta^n \to P}$. It gives rise to a map $\phi^* : A_{PL}(P') \to A_{PL}(P)$ of commutative dg algebras. Let T be a topological space. If we write $S_*(T)$ for the singuar simplicial complex whose n-th term is the set of singular n-simplices, the commutative dg algebra $A_{PL}(T)$ is defined to be $A_{PL}(S_*(T))$.

The assignment $P \mapsto A_{PL}(P)$ gives rise to a functor $A_{PL} : \mathrm{Set}_\Delta \to (\mathrm{CAlg}^{dg}_{\mathbb{Q}})^{op}$ to the category $\mathrm{CAlg}^{dg}_{\mathbb{Q}}$ of commutative dg algebras over $\mathbb{Q}$. There exists a canonical equivalence between the ∞-category $\mathcal{S}$ of spaces and the ∞-category obtained from Set_Δ by inverting weak homotopy equivalences (cf. [33, 1.3.4.21]). As observed below, the functor A_{PL} sends a weak homotopy equivalence in Set_Δ to a quasi-isomorphism in $\mathrm{CAlg}^{dg}_{\mathbb{Q}}$. Therefore, $A_{PL} : \mathrm{Set}_\Delta \to (\mathrm{CAlg}^{dg}_{\mathbb{Q}})^{op}$ induces

$$A_{PL,\infty} : \mathcal{S} \longrightarrow \mathrm{N}(\mathrm{CAlg}^{dg}_{\mathbb{Q}})[W^{-1}]^{op} \simeq \mathrm{CAlg}^{op}_{\mathbb{Q}} .$$

For a topological space T, we shall denote by $A_{PL,\infty}(T)$ the image of $A_{PL}(T)$ in $\mathrm{CAlg}_{\mathbb{Q}}$.

First we will describe the induced functor $A_{PL,\infty} : \mathcal{S} \to \mathrm{CAlg}^{op}_{\mathbb{Q}}$ in an intrinsic way.

Proposition 4.1. *The following conditions hold:*

(1) The functor $A_{PL} : \mathrm{Set}_\Delta \to \mathrm{CAlg}^{dg}_{\mathbb{Q}}$ sends a weak homotopy equivalence in Set_Δ to a quasi-isomorphism in $\mathrm{CAlg}^{dg}_{\mathbb{Q}}$,

(2) $A_{PL,\infty}(\Delta^0) \simeq \mathbb{Q}$,

(3) $A_{PL,\infty} : \mathcal{S} \to \mathrm{CAlg}^{op}_{\mathbb{Q}}$ preserves small colimits.

Remark 4.2. The functor $A_{PL,\infty}$ is uniquely determined by the properties (2) and (3) in Proposition 4.1. Let $\mathrm{Fun}^{\mathrm{L}}(\mathcal{S}, \mathrm{CAlg}^{op}_{\mathbb{Q}})$ be the full subcategory of $\mathrm{Fun}(\mathcal{S}, \mathrm{CAlg}^{op}_{\mathbb{Q}})$ spanned by those functors that preserve small colimits. Then by left Kan extension [32, 5.1.5.6], the map $p : \Delta^0 \to \mathcal{S}$ with value Δ^0 (i.e. the contractible space) induces an equivalence

$$\mathrm{Fun}^{\mathrm{L}}(\mathcal{S}, \mathrm{CAlg}^{op}_{\mathbb{Q}}) \xrightarrow{\sim} \mathrm{Fun}(\Delta^0, \mathrm{CAlg}^{op}_{\mathbb{Q}}) \simeq \mathrm{CAlg}^{op}_{\mathbb{Q}} .$$

Therefore, the colimit-preserving functor $A_{PL,\infty}$ is uniquely determined by the value $\mathbb{Q}$ of the contractible space. Namely, if $u : \Delta^0 \to \mathrm{CAlg}^{op}_{\mathbb{Q}}$ denotes the map determined by the object $\mathbb{Q}$ of $\mathrm{CAlg}_{\mathbb{Q}}$, then $A_{PL,\infty} : \mathcal{S} \to \mathrm{CAlg}^{op}_{\mathbb{Q}}$ is a left Kan extension of $u : \Delta^0 \to \mathrm{CAlg}^{op}_{\mathbb{Q}}$ along $p : \Delta^0 \to \mathcal{S}$.

Proof. We first prove (1). Let $\mathrm{CAlg}^{dg}_{\mathbb{Q}} \to \mathrm{Comp}(\mathbb{Q})$ be the forgetful functor to the category $\mathrm{Comp}(\mathbb{Q})$ of complexes of $\mathbb{Q}$-vector spaces. It is enough to show that the composition $\mathrm{Set}_\Delta \to (\mathrm{CAlg}^{dg}_{\mathbb{Q}})^{op} \to \mathrm{Comp}(\mathbb{Q})^{op}$ preserves quasi-isomorphisms. According to [16, Theorem 10.9], there is the zig-zag of quasi-isomorphisms in $\mathrm{Comp}(\mathbb{Q})$

$$C^*(P) \to B(P) \leftarrow A_{PL}(P)$$

where $C^*(P)$ is the cochain complex associated to a simplicial set P with rational coefficients, and $B(P)$ is a certain "intermediate" cochain complex associated to P. These quasi-isomorphisms are functorial in the sense that for any map $P \to P'$ of simplicial sets, they commute with $A_{PL}(P') \to A_{PL}(P)$, $B(P') \to B(P)$, and $C^*(P') \to C^*(P)$. Thus, it will suffice to observe that $C^* : \mathrm{Set}_\Delta \to \mathrm{Comp}(\mathbb{Q})^{op}$ given by $P \mapsto C^*(P)$ sends weak homotopy equivalences to quasi-isomorphisms. Let $C_* : \mathrm{Set}_\Delta \to \mathrm{Comp}(\mathbb{Q})$ be the functor which carries P to the (normalized) chain complex $C_*(P)$ with rational coefficients. Since the dual of any quasi-isomorphism $C_*(P) \to C_*(P')$ is a quasi-isomorphism $C^*(P') \to C^*(P)$, we are reduced to proving that C_* sends weak homotopy equivalences to quasi-isomorphisms. Indeed, it is a well-known fact, but we here describe one of the proofs. Let Vect_Δ denote the category of simplicial objects in the category of $\mathbb{Q}$-vector spaces, that is, simplicial $\mathbb{Q}$-vector spaces. Consider the adjunction $\mathbb{Q}[-] : \mathrm{Set}_\Delta \rightleftarrows \mathrm{Vect}_\Delta : U$ where U is the forgetful functor, and $\mathbb{Q}[-]$ is its left adjoint, that is, the free functor. Let us consider Set_Δ as the Quillen model category whose weak equivalences are weak homotopy equivalence, and whose cofibrations are monomorphisms. As in the case of simplicial abelian groups, Vect_Δ

admits a model category structure in which f is a weak equivalences (resp. a fibration) if $U(f)$ is a weak equivalence (resp. a Kan fibration). Then the pair $(\mathbb{Q}[-], U)$ is a Quillen adjunction. Let $N : \mathrm{Vect}_\Delta \xrightarrow{\sim} \mathrm{Comp}^{\le 0}(\mathbb{Q})$ be the Dold-Kan equivalence which carries a simplicial vector space to its normalized chain complex, where $\mathrm{Comp}^{\le 0}(\mathbb{Q})$ is the full subcategory of $\mathrm{Comp}(\mathbb{Q})$ spanned by those object C such that $H^i(C) = 0$ for $i > 0$. The composite $\mathrm{Set}_\Delta \xrightarrow{\mathbb{Q}[-]} \mathrm{Vect}_\Delta \xrightarrow{N} \mathrm{Comp}^{\le 0}(\mathbb{Q})$ is naturally equivalent to the functor $\mathrm{Set}_\Delta \to \mathrm{Comp}^{\le 0}(\mathbb{Q})$ which sends P to $C_*(P)$. The functor $\mathbb{Q}[-]$ preserves weak equivalences since every object in Set_Δ is cofibrant, and N sends weak equivalences to quasi-isomorphims. Thus, C_* sends weak homotopy equivalences to quasi-isomorphisms.

The equality $A_{PL}(\Delta^0) = \mathbb{Q}$ is clear from the definition (see [16, Example 1 in page 124]). Hence (2) follows.

Next we prove (3). Note that the forgetful functor $\mathrm{CAlg}_\mathbb{Q} \to \mathrm{Mod}_\mathbb{Q} \simeq \mathsf{D}(\mathbb{Q})$ preserves limits (cf. [33, 3.2.2.4]). Thus, it will suffice to prove that $\mathcal{S} \xrightarrow{A_{PL,\infty}} \mathrm{CAlg}_\mathbb{Q}^{op} \to \mathrm{Mod}_\mathbb{Q}^{op}$ preserves small colimits; a small colimit diagram in $\mathcal{S}$ maps to a limit diagram in $\mathrm{Mod}_\mathbb{Q}$. According to [32, 4.4.2.7], $\mathcal{S} \to \mathrm{Mod}_\mathbb{Q}^{op}$ preserves small colimits if and only if it preserves pushouts and small coproducts. It is enough to show that $\mathrm{Set}_\Delta \xrightarrow{A_{PL}} (\mathrm{CAlg}_\mathbb{Q}^{dg})^{op} \to \mathrm{Comp}(\mathbb{Q})^{op}$ sends homotopy pushout diagrams and homotopy coproduct diagrams to homotopy pullback diagrams and homotopy product diagrams in $\mathrm{Comp}(\mathbb{Q})$, respectively. Here the second functor is the forgetful functor, and $\mathrm{Comp}(\mathbb{Q})$ is endowed with the projective model structure, see Section 2. As discussed in the proof of (1), we may replace this composite by $C^* : \mathrm{Set}_\Delta \to \mathrm{Comp}(\mathbb{Q})^{op}$. We will observe that $C_* : \mathrm{Set}_\Delta \to \mathrm{Comp}(\mathbb{Q})$ preserves homotopy colimits. We equip $\mathrm{Comp}^{\le 0}(\mathbb{Q})$ with the projective model structures (cf. [23, 2.3], [46, 4.1]). A morphism p in $\mathrm{Comp}^{\le 0}(\mathbb{Q})$ is a weak equivalence (resp. a fibration) if it is a quasi-isomorphism (resp. surjective in cohomologically negative degrees). Cofibrations are monomorphisms (keep in mind that $\mathbb{Q}$ is a field). The free functor $\mathbb{Q}[-] : \mathrm{Set}_\Delta \to \mathrm{Vect}_\Delta$ is a left Quillen functor. The normalization functor $N : \mathrm{Vect}_\Delta \to \mathrm{Comp}^{\le 0}(\mathbb{Q})$ is a left Quillen functor (see [46, Section 4]). In addition, $\mathrm{Comp}^{\le 0}(\mathbb{Q}) \hookrightarrow \mathrm{Comp}(\mathbb{Q})$ is a left Quillen functor. Therefore, we deduce that $C_* : \mathrm{Set}_\Delta \to \mathrm{Comp}(\mathbb{Q})$ preserves homotopy colimits. Note that C^* is composite $\mathrm{Set}_\Delta \xrightarrow{C_*} \mathrm{Comp}(\mathbb{Q}) \to \mathrm{Comp}(\mathbb{Q})^{op}$ where the second functor is given by the hom complex $\mathcal{H}om_\mathbb{Q}(-, \mathbb{Q})$. Then $\mathcal{H}om_\mathbb{Q}(-, \mathbb{Q}) : \mathrm{Comp}(\mathbb{Q}) \to \mathrm{Comp}(\mathbb{Q})^{op}$ preserves homotopy colimits, so that the induced functor $\mathrm{Mod}_\mathbb{Q} \to \mathrm{Mod}_\mathbb{Q}^{op}$ preserves colimits. (Indeed, it is enough to check that it preserves homotopy pushouts and homotopy coproducts. In $\mathrm{Comp}(\mathbb{Q})$, every object is both cofibrant and fibrant. By the explicit presentation of homotopy pushouts/coproducts cf. [32, A.2.4.4], we easily see that $\mathcal{H}om_\mathbb{Q}(-, \mathbb{Q})$ sends a homotopy pushout (resp. coproduct) diagram to a homotopy pullback (resp. coproduct) diagram.) Consequently, $\mathcal{S} \to \mathrm{Mod}_\mathbb{Q}^{op}$ induced by C^* preserves small colimits. □

4.3 Let us consider the composite

$$T : \mathrm{Sm}_k^{op} \xrightarrow{\Xi} \mathsf{CAlg}(\mathsf{DM}^\otimes(k)) \xrightarrow{\mathsf{R}} \mathrm{CAlg}_\mathbb{Q}$$

See Proposition 3.4 for Ξ. We put $T_X = T(X) = \mathsf{R}(M_X)$.

Theorem 4.3. *Let X be a smooth scheme separated of finite type over $k \subset \mathbb{C}$. Let X^t be the underlying topological space of the complex manifold $X \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, \mathbb{C}$. There is a canonical equivalence $\mathsf{R}(M_X) = T_X \xrightarrow{\sim} A_{PL,\infty}(X^t)$ in $\mathrm{CAlg}_\mathbb{Q}$.*

Proof. We first introduce some categories. Let $\mathsf{D}^{\otimes}(X^t)$ be the symmetric monoidal presentable ∞-category of complexes of sheaves of $\mathbb{Q}$-vector spaces on X^t. We define this ∞-category by the machinery of model categories. Let $\mathrm{Sh}(X^t)$ be the Grothendieck abelian category of sheaves of $\mathbb{Q}$-vector spaces on X^t and let $\mathrm{Comp}(\mathrm{Sh}(X^t))$ be the category of cochain complexes of $\mathrm{Sh}(X^t)$. It is endowed with the symmetric monoidal structure by tensor product. Thanks to [11, Theorem 2.5, Example 2.3, Proposition 3.2], there is a symmetric monoidal combinatorial model category structure on $\mathrm{Comp}(\mathrm{Sh}(X^t))$ in which weak equivalences consists of quasi-isomorphisms. We then obtain the symmetric monoidal presentable ∞-category $\mathsf{D}^{\otimes}(X^t)$ from $\mathrm{Comp}(\mathrm{Sh}(X^t))^c$ by inverting weak equivalences (cf. Section 2). By replacing X^t with the one-point space $*$, we also have a symmetric monoidal combinatorial model category $\mathrm{Comp}(\mathrm{Sh}(*))$ which coincides with $\mathrm{Comp}(\mathbb{Q})$ endowed with the projective model structure. By abuse of notation we denote the associated symmetric monoidal presentable ∞-category by $\mathsf{D}^{\otimes}(\mathbb{Q})$. The canonical map to the one-point space $f^t : X^t \to *$ induces the symmetric monoidal pullback functor $\mathrm{Comp}(\mathrm{Sh}(*)) \to \mathrm{Comp}(\mathrm{Sh}(X^t))$ that is a left Quillen functor [11, Theorem 2.14]. It gives rise to a symmetric monoidal colimit-preserving pullback functor $f^{t*} : \mathsf{D}(\mathbb{Q}) \to \mathsf{D}(X^t)$. According to relative adjoint functor theorem [33, 7.3.2.6], there is a right adjoint functor $f^t_* : \mathsf{D}(X^t) \to \mathsf{D}(\mathbb{Q})$ which is lax symmetric monoidal. We then use the Beilinson motives studied by Cisinski-Déglise [13]. Let $\mathbb{M}_B(X)$ be the symmetric monoidal combinatorial model category of Beilinson motives over X with rational coefficients (see [13, 14.2]) and let $\mathsf{DM}^{\otimes}_B(X)$ be the symmetric monoidal presentable ∞-category obtained from $\mathbb{M}_B(X)$. Since X is regular, according to [13, 16.1.1, 16.1.4] there is a symmetric monoidal equivalence $\mathsf{DM}^{\otimes}_B(X) \xrightarrow{\sim} \mathsf{DM}^{\otimes}(X)$ induced by a symmetric monoidal left Quillen functor $\mathbb{M}_B(X) \to Sp_{\mathrm{Tate}}(X)$ (by [13, 16.1.4] the induced functor between their homotopy categories is an equivalence, from which the equivalence of stable ∞-categories follows, see e.g. [24, Lemma 5.8]). The equivalences $\mathsf{DM}^{\otimes}_B(X) \simeq \mathsf{DM}^{\otimes}(X)$ and $\mathsf{DM}^{\otimes}_B(\operatorname{Spec} k) \simeq \mathsf{DM}^{\otimes}(k)$ commute with pullback functors. Let $\mathsf{R}_X : \mathsf{DM}^{\otimes}(X) \simeq \mathsf{DM}^{\otimes}_B(X) \to \mathsf{D}^{\otimes}(X^t)$ be the (relative) realization functor that is a symmetric monoidal functor. It is obtained from symmetric monoidal functors of model categories (cf. Section 2): as explained in [13, 17.1.7] that uses the construction of Ayoub, there is a diagram of symmetric monoidal functors $\mathbb{M}_B(X) \xrightarrow{r} \mathbb{M}(X) \xleftarrow{p} \mathrm{Comp}(\mathrm{Sh}(X^t))$ of model categories where $\mathbb{M}(X)$ is an intermediate symmetric monoidal model category, r is a symmetric monoidal left Quillen functor, and p induces an equivalence of symmetric monoidal ∞-categories. Similarly, we have the realization functor $\mathsf{R} : \mathsf{DM}^{\otimes}(k) \simeq \mathsf{DM}^{\otimes}_B(\operatorname{Spec} k) \to \mathsf{D}^{\otimes}(\mathbb{Q})$ of singular cohomology theory. The functors R and R_X commute with the pullback functors (because of the construction). Therefore, we have the diagram

$$\begin{array}{ccc} \mathsf{DM}^{\otimes}(X) & \xrightarrow{\mathsf{R}_X} & \mathsf{D}^{\otimes}(X^t) \\ {\scriptstyle f^*}\uparrow\downarrow{\scriptstyle f_*} & & {\scriptstyle f^t_*}\downarrow\uparrow{\scriptstyle f^{t*}} \\ \mathsf{DM}^{\otimes}(k) & \xrightarrow[\mathsf{R}]{} & \mathsf{D}^{\otimes}(\mathbb{Q}). \end{array}$$

with a canonical equivalence $\mathsf{R}_X \circ f^* \simeq f^{t*} \circ \mathsf{R}$ of symmetric monoidal functors. Let A be a commutative algebra object in $\mathsf{DM}^{\otimes}(X)$, that is, an object of $\mathrm{CAlg}(\mathsf{DM}^{\otimes}(X))$. Consider the canonical exchange map $e : \mathsf{R}(f_*(A)) \to f^t_*(\mathsf{R}_X(A))$ in $\mathsf{D}(\mathbb{Q})$. This map is the composition of morphisms

$$\begin{array}{rcl} \mathsf{R}(f_*(A)) & \to & f^t_* f^{t*}(\mathsf{R}(f_*(A)) \\ & \simeq & f^t_* \mathsf{R}_X f^*(f_*(A)) \\ & \simeq & f^t_* \mathsf{R}_X (f^* f_*)(A) \\ & \to & f^t_* \mathsf{R}_X(A) \end{array}$$

where the first arrow is induced by the unit map $\mathrm{id} \to f^t_* f^{t*}$, the second arrow is induced by $\mathsf{R}_X f^* \simeq f^{t*}\mathsf{R}$, and the fourth one is induced by the counit map $f^* f_* \to \mathrm{id}$. The unit map $\mathrm{id} \to f^t_* f^{t*}$ and the counit map $f^* f_* \to \mathrm{id}$ are promoted to a unit map and a counit map for adjunctions $\mathrm{CAlg}(\mathsf{D}^{\otimes}(\mathbb{Q})) \rightleftarrows \mathrm{CAlg}(\mathsf{D}^{\otimes}(X^t))$ and $\mathrm{CAlg}(\mathsf{DM}^{\otimes}(k)) \rightleftarrows \mathrm{CAlg}(\mathsf{DM}^{\otimes}(X))$, respectively. In particular, $e : \mathsf{R}(f_*(A)) \to f^t_*(\mathsf{R}_X(A))$ is promoted to a morphism in $\mathrm{CAlg}(\mathsf{D}^{\otimes}(\mathbb{Q})) \simeq \mathrm{CAlg}_{\mathbb{Q}}$. By [13, 17.2.18, 4.4.25], if A is compact in the underlying ∞-category $\mathsf{DM}(X)$, e is an equivalence. In particular, if $A = \mathbf{1}_X$, we have a canonocal equivalence $\mathsf{R}(f_*(\mathbf{1}_X)) = \mathsf{R}(M_X) \xrightarrow{\sim} f^t_*(\mathsf{R}_X(\mathbf{1}_X))$ in $\mathrm{CAlg}_{\mathbb{Q}}$. Consequently, to prove our assertion it suffices to prove that $f^t_*(\mathbf{1}_{X^t})$ is equivalent to $A_{PL,\infty}(X^t)$ where $\mathbf{1}_{X^t}$ is the unit of $\mathsf{D}^{\otimes}(X^t)$, i.e., the constant sheaf with value $\mathbb{Q}$.

For this purpose, recall first that since $X \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, \mathbb{C}$ is a complex smooth scheme separated of finite type, the underlying topological space X^t is a hausdorff paracompact smooth manifold. Therefore, according to [9, Theorem 5.1], it admits a good cover $\mathcal{U} = \{U_\lambda\}_{\lambda \in I}$, that is, an open cover $\mathcal{U} = \{U_\lambda\}_{\lambda \in I}$ such that every non-empty finite intersection $U_{\lambda_0} \cap \ldots \cap U_{\lambda_r}$ is contractible. Take the augmented simplicial diagram of the Cech nerve $U_\bullet \to U_{-1} := X^t$ associated to the cover. The n-th term U_n of $U_\bullet$ is the disjoint union of intersections of $n+1$ open sets in $\mathcal{U}$. We denote by $j_{U_n} : U_n \to X^t = U_{-1}$ the canonical map. If we think of $U_\bullet \to X^t$ as an augmented simplicial diagram in $\mathcal{S}$, then by Dugger-Isaksen [15, Theorem 1.1], it is a colimit diagram. According to Proposition 4.1, the functor $A_{PL,\infty} : \mathcal{S} \to \mathrm{CAlg}_{\mathbb{Q}}^{op}$ commutes with small colimits. Thus, the canonical morphism $A_{PL,\infty}(X^t) \to \varprojlim_{[n]\in\Delta} A_{PL,\infty}(U_n)$ is an equivalence where $\varprojlim_{[n]\in\Delta} A_{PL,\infty}(U_n)$ is a limit of the cosimplicial diagram in $\mathrm{CAlg}_{\mathbb{Q}}$. Thus, it is enough to show that $\varprojlim_{[n]\in\Delta} A_{PL,\infty}(U_n) \simeq f^t_*(\mathbf{1}_{X^t})$. For $i \geq -1$, we let $\mathrm{Comp}(\mathrm{Sh}(U_n))$ be the category of complexes of sheaves of $\mathbb{Q}$-vector spaces on U_n. As in the case of $\mathsf{D}(X^t)$, by the model structure in [11, 2.3, 2.5] we have a symmetric monoidal presentable ∞-category $\mathsf{D}^{\otimes}(U_n)$ from $\mathrm{Comp}(\mathrm{Sh}(U_n))$. For each morphism $U_n \to U_m$, a symmetric monoidal colimit-preserving functor $\mathsf{D}^{\otimes}(U_m) \to \mathsf{D}^{\otimes}(U_n)$. It gives rise to a cosimplicial diagram of symmetric monoidal ∞-categories which we denote simply by $\mathsf{D}^{\otimes}(U_\bullet)$. It also has the natural coaugmentation $\mathsf{D}^{\otimes}(X^t) \to \mathsf{D}^{\otimes}(U_\bullet)$. Let $\Gamma(U_n, -) : \mathsf{D}(U_n) \to \mathsf{D}(\mathbb{Q})$ be the (derived) global section functor, that is a lax symmetric monoidal right adjoint functor to the pullback functor $\mathsf{D}^{\otimes}(\mathbb{Q}) \to \mathsf{D}^{\otimes}(U_n)$ of $U_n \to *$. We denote by $\mathbf{1}_{U_n}$ the unit of $\mathsf{D}(U_n)$ that corresponds to the constant sheaf with value $\mathbb{Q}$. Note $f^t_*(-) = \Gamma(U_{-1}, -) = \Gamma(X^t, -)$, and $\Gamma(U_n, \mathbf{1}_{U_n})$ in $\mathsf{D}(\mathbb{Q})$ is a complex computing the sheaf cohomology of U_n with coefficients in $\mathbb{Q}$. Remember that U_n is a disjoint union of contractible spaces for $n \geq 0$. For each connected component V of U_n, $\Gamma(V, \mathbf{1}_{U_n}|_V)$ in $\mathrm{CAlg}_{\mathbb{Q}}$ is an initial object of $\mathrm{CAlg}_{\mathbb{Q}}$, i.e., $\mathbb{Q}$ since the unit map $\mathbb{Q} \to \Gamma(V, \mathbf{1}_{U_n}|_V)$ is an equivalence in $\mathsf{D}(\mathbb{Q})$, cf. [33, Corollary 3.2.1.9]. By Proposition 4.1, the image of a contractible space under $A_{PL,\infty}$ is $\mathbb{Q}$. Therefore, $\Gamma(U_n, \mathbf{1}_{U_n}) \in \mathrm{CAlg}_{\mathbb{Q}}$ is equivalent to $A_{PL,\infty}(U_n)$, i.e., $\Gamma(U_n, \mathbf{1}_{U_n}) \simeq \prod_{\pi_0(U_n)} \mathbb{Q} \simeq A_{PL,\infty}(U_n)$ for $n \geq 0$ ($\pi_0(-)$ is the set of connected components). We may consider $\{\Gamma(U_n, \mathbf{1}_{U_n})\}_{[n]\in\Delta}$ to be a cosimplicial diagram of ordinary commutative algebras (arising from connected components of $U_\bullet$). We then have $\varprojlim A_{PL,\infty}(U_n) \simeq \varprojlim \Gamma(U_n, \mathbf{1}_{U_n})$. It will suffice to prove that the canonical morphism $\Gamma(X^t, \mathbf{1}_{X^t}) \to \varprojlim_{[n]\in\Delta} \Gamma(U_n, \mathbf{1}_{U_n})$ in $\mathsf{D}(\mathbb{Q})$ is an equivalence (we may and will disregard their commutative algebra structures). To this end, we use the descent for hypercovers on X^t. Let $j_{U_n!}:\mathsf{D}(U_n)\to\mathsf{D}(X^t)$ be the left adjoint to the restriction $j^*_{U_n}:\mathsf{D}(X^t)\to\mathsf{D}(U_n)$. According to [11, Example 2.3, Theorem 2.5], we see that the canonical morphism $\varinjlim_{[n]\in\Delta^{op}} j_{U_n!}(\mathbf{1}_{U_n})\to\mathbf{1}_{X^t}$ is an equivalence in $\mathsf{D}(X^t)$. For any F in $\mathsf{D}(X^t)$, it induces an equivalence $\Gamma(X^t,F)\xrightarrow{\sim}\varprojlim_{[n]\in\Delta}\Gamma(U_n,F|_{U_n})$. In particular, we have a canonical equivalence $\Gamma(X^t,\mathbf{1}_{X^t})\xrightarrow{\sim}\varprojlim_{[n]\in\Delta}\Gamma(U_n,\mathbf{1}_{U_n})$. □

Remark 4.4. Let $\phi:Y\to X$ be a morphism in Sm_k. Then $\phi^*:M_X\to M_Y$ induces $\mathsf{R}(\phi^*):\mathsf{R}(M_X)=T_X\to\mathsf{R}(M_X)=T_Y$. On the other hand, the associated continuous map $\phi^t:Y^t\to X^t$ of topological spaces induces $\phi^{t*}:A_{PL,\infty}(X^t)\to A_{PL,\infty}(Y^t)$ induced by $A_{PL}(X^t)\to A_{PL}(Y^t)$. The morphism $\mathsf{R}(\phi^*):T_X\to T_Y$ in $\mathrm{CAlg}_{\mathbb{Q}}$ is equivalent to $\phi^{t*}:A_{PL,\infty}(X^t)\to A_{PL,\infty}(Y^t)$ through equivalences $T_X\simeq A_{PL,\infty}(X^t)$ and $T_Y\simeq A_{PL,\infty}(Y^t)$ in Theorem 4.3.

To observe this, note first that by the compatibility of the realization functor with push-forward functors, $\mathsf{R}(\phi^*)$ can be identified with $f^t_*(\mathbf{1}_{X^t})\to g^t_*(\mathbf{1}_{Y^t})$ induced by $\phi^t:Y^t\to X^t$ where $g^t:Y^t\to *$ is the canonical map to one point space. Let us unfold the equaivalence given in the proof of Theorem 4.3. As in the proof, choose a good cover $\mathcal{U}=\{U_\lambda\}_{\lambda\in I}$ of X^t and take the augmented Cech nerve $U_\bullet\to X^t=U_{-1}$. We know from the proof of Theorem 4.3 that there are canonical equivalences $\Gamma(U_n,\mathbf{1}_{U_n})\xrightarrow{\sim}\prod_{\alpha\in\pi_0(U_n)}\Gamma(U_{n,\alpha},\mathbf{1}_{U_{n,\alpha}})\xleftarrow{\sim}\prod_{\alpha\in\pi_0(U_n)}\mathbb{Q}$ in $\mathrm{CAlg}_{\mathbb{Q}}$ where each U_n is a disjoint union $\sqcup_{\alpha\in\pi_0(U_n)}U_{n,\alpha}$ of contractible spaces. Similarly, we have canonical equivalences $A_{PL,\infty}(U_n)\xrightarrow{\sim}\prod_{\alpha\in\pi_0(U_n)}A_{PL,\infty}(U_{n,\alpha})\xleftarrow{\sim}\prod_{\alpha\in\pi_0(U_n)}\mathbb{Q}$. Both objects $\{\Gamma(U_n,\mathbf{1}_{U_n})\}_{[n]\in\Delta}$ and $\{A_{PL}(U_n)\}_{[n]\in\Delta}$ are equivalent to the cosimplicial ordinary commutative $\mathbb{Q}$-algebra, regarded as a cosimplicial object in $\mathrm{CAlg}_{\mathbb{Q}}$, that is defined by the assignment $[n]\mapsto\prod_{\alpha\in\pi_0(U_n)}\mathbb{Q}=\mathbb{Q}^{\pi_0(U_n)}$ such that for any $[n]\to[m]$, $\mathbb{Q}^{\pi_0(U_n)}\to\mathbb{Q}^{\pi_0(U_m)}$ is induced by the map $\pi_0(U_m)\to\pi_0(U_n)$ (by the superscript we mean cotensor). It gives rise to $A_{PL,\infty}(X^t)\simeq\varprojlim A_{PL,\infty}(U_n)\simeq\varprojlim\Gamma(U_n,\mathbf{1}_{U_n})\simeq f^t_*(\mathbf{1}_{X^t})$. Taking into account these steps, we are reduced to checking a functoriality of good covers: it suffices to verify that if $\mathcal{U}=\{U_\lambda\}_{\lambda\in I}$ is a good cover of X^t, then there is a good over $\mathcal{V}=\{V_\mu\}_{\mu\in J}$ of Y^t such that any $V_\mu\to Y^t\to X^t$ factors through some $U_\lambda\to X^t$. Actually, it follows from the proof of the existence of a good cover. See [9, Corollary 5.2] and the discussion after the proof of [9, Theorem 5.1].

It is useful to have a smooth de Rham model of T_X. We will describe $T_X\otimes_{\mathbb{Q}}\mathbb{R}$ in terms of smooth differential forms. By X_∞ we mean the underlying differential manifold of $X\times_{\mathrm{Spec}\,k}\mathrm{Spec}\,\mathbb{C}$. Let A_{X_∞} be the commutative dg algebra of C^∞ real differential forms on X_∞. We call A_{X_∞} the smooth de Rham algebra on X_∞ We think of A_{X_∞} as an object in $\mathrm{CAlg}_{\mathbb{R}}$.

Corollary 4.5. *Consider the base change $T_X\otimes_{\mathbb{Q}}\mathbb{R}$ which belongs to $\mathrm{CAlg}_{\mathbb{R}}$. There is an equivalence $T_X\otimes_{\mathbb{Q}}\mathbb{R}\simeq A_{X_\infty}$ in $\mathrm{CAlg}_{\mathbb{R}}$.*

Proof. There is a zig-zag of quasi-isomorphisms between A_{X_∞} and $A_{PL}(X^t)\otimes_{\mathbb{Q}}\mathbb{R}$ (see [16, Theorem 11.4]). Thus, by Theorem 4.3 we see that $T_X\otimes_{\mathbb{Q}}\mathbb{R}\simeq A_{X_\infty}$. □

By using Theorem 4.3 and Remark 4.4, we can easily prove the following:

Proposition 4.6. *Let $\mathrm{CAlg}(\mathsf{DM}^{\otimes}(k))\to\mathrm{CAlg}_{\mathbb{Q}}$ be the multiplicative realization functor. Then the image of the motivic algebra of path torsor $P(X,x,y)$ (cf. Example 3.12) in $\mathrm{CAlg}_{\mathbb{Q}}$ is equivalent to the pushout $\mathbb{Q}\otimes_{A_{PL,\infty}(X^t)}\mathbb{Q}$ associated to two augmentations $A_{PL,\infty}(X^t)\to\mathbb{Q}$ and $A_{PL,\infty}(X^t)\to\mathbb{Q}$ respectively induced by points x and y in X^t. (We remark that $\mathbb{Q}\otimes_{A_{PL,\infty}(X)}\mathbb{Q}$ can be obtained by a bar construction of $A_{PL}(X)$ with two augmentations, see [41].)*

The image of HHM_X (cf. Example 3.13) in $\mathrm{CAlg}_{\mathbb{Q}}$ *is*

$$A_{PL,\infty}(X^t) \otimes_{A_{PL,\infty}(X^t)\otimes A_{PL,\infty}(X^t)} A_{PL,\infty}(X^t) \simeq S^1 \otimes A_{PL,\infty}(X^t).$$

(It might be worth mentioning that if X^t is simply connected, then

$$A_{PL,\infty}(X^t) \otimes_{A_{PL,\infty}(X^t)\otimes A_{PL,\infty}(X^t)} A_{PL,\infty}(X^t)$$

is equivalent to $A_{PL,\infty}(LX^t)$ where LX^t is the free loop space of X^t [16, Example 1 in page 206].)

4.4 Before proceeding to the next subsection, we introduce some algebro-geometric notions. Let K be a field of characteristic zero. Let $\mathrm{CAlg}_K^{\mathrm{dis}}$ be the full subcategory of CAlg_K that is spanned by discrete objects C, i.e., $H^i(C) = 0$ for $i \neq 0$. Put another way, we let $\mathrm{Mod}_K^{\mathrm{dis}}$ be the (symmetric monoidal) full subcategory of $\mathrm{Mod}_K \simeq \mathsf{D}(K)$ spanned by discrete objects M, i.e., $H^i(M) = 0$ for $i \neq 0$. This full subcategory is nothing else but (the nerve of) the category of K-vector spaces. Then $\mathrm{CAlg}_K^{\mathrm{dis}} = \mathrm{CAlg}(\mathrm{Mod}_K^{\mathrm{dis}})$. The ∞-category $\mathrm{CAlg}_K^{\mathrm{dis}}$ is naturally equivalent to the nerve of category of ordinary commutative K-algebras. Let Aff_K be the opposite category of CAlg_K. We write $\mathrm{Spec}\, R$ for an object in Aff_K that corresponds to $R \in \mathrm{CAlg}_K$. We shall refer to it as a derived affine scheme (or affine scheme) over K. The Yoneda embedding identifies Aff_K with a full subcategory of $\mathrm{Fun}(\mathrm{CAlg}_K, \mathcal{S})$. This embedding preserves small limits. The functor $\mathrm{Spec}\, R : \mathrm{CAlg}_K \to \mathcal{S}$ corepresented by R satisfies the sheaf condition with respect to the flat topology, see e.g. [34]. We often regard $\mathrm{Spec}\, R$ as a sheaf $\mathrm{CAlg}_K \to \mathcal{S}$. We remark that in the literature of derived geometry (see e.g. [34] for its ∞-categorical theory), $\mathrm{Spec}\, R$ with $R \in \mathrm{CAlg}_K$ is usually called a nonconnective (derived) affine scheme. Let $\mathrm{Aff}_K^{\mathrm{dis}}$ be the full subcategory of Aff_K that corresponds to $\mathrm{CAlg}_K^{\mathrm{dis}}$. One can naturally identify $\mathrm{Aff}_K^{\mathrm{dis}}$ with the category of ordinary affine schemes over K (keep in mind that the full subcategories $\mathrm{Aff}_K^{\mathrm{dis}}$ are not closed under some constructions; for example, in general, fiber products in $\mathrm{Aff}_K^{\mathrm{dis}}$ are not compatible with those in Aff_K).

For an ∞-category $\mathcal{C}$ that has finite products, we write $\mathrm{Grp}(\mathcal{C})$ for the ∞-category of group objects in $\mathcal{C}$. We shall call a group object in Aff_K a derived affine group scheme over K. There is a canonical Yoneda embedding $\mathrm{Grp}(\mathrm{Aff}_K) \hookrightarrow \mathrm{Fun}(\mathrm{CAlg}_K, \mathrm{Grp}(\mathcal{S}))$. Therefore, through this functor we often think of a derived affine group scheme as a sheaf $\mathrm{CAlg}_K \to \mathrm{Grp}(\mathcal{S})$. Put another way, $\mathrm{Spec}\, R$ in $\mathrm{Grp}(\mathrm{Aff}_K)$ amounts to a commutative Hopf algebra object R in $\mathrm{Mod}_K^{\otimes}$. See [24, Appendix A] for details.

4.5

Definition 4.7. In Section 3.5, for a pointed smooth variety (X, x) and a natural number $n \geq 1$, we have defined the group object $\mathcal{G}^{(n)}(X, x) : \mathrm{N}(\Delta^{op}) \to \mathrm{CAlg}(\mathsf{DM}^{\otimes}(k))^{op}$. Since the multiplicative realization functor $\mathrm{CAlg}(\mathsf{R}_E) : \mathrm{CAlg}(\mathsf{DM}^{\otimes}(k)) \to \mathrm{CAlg}_K$ preserves coproducts, we see that the composite

$$G_E^{(n)}(X, x) : \mathrm{N}(\Delta^{op}) \stackrel{\mathcal{G}^{(n)}(X,x)^{op}}{\longrightarrow} \mathrm{CAlg}(\mathsf{DM}^{\otimes}(k))^{op} \stackrel{\mathrm{CAlg}(\mathsf{R}_E)^{op}}{\longrightarrow} \mathrm{CAlg}_K^{op} = \mathrm{Aff}_K$$

is a group object in Aff_K. Namely, $G_E^{(n)}(X, x)$ is a derived affine group scheme over K. If no confusion is likely to arise, we often write $G^{(n)}(X, x)$ for $G_E^{(n)}(X, x)$.

Proposition 4.8. *Suppose that k is embedded in $\mathbb{C}$ and consider the case of singular realization $\mathsf{R} = \mathsf{R}_E$. A closed point x on X determines a point of the associated topological space X^t which we denote also by x. Let $\operatorname{Spec}\mathbb{Q} \to \operatorname{Spec} T_X$ be a morphism induced by x. Then the derived affine group scheme $G(X,x) = G^{(1)}(X,x)$ is equivalent to the Cech nerve obtained from $\operatorname{Spec}\mathbb{Q} \to \operatorname{Spec} T_X$. The iterated group scheme $G^{(n)}(X,x)$ ($n \geq 2$) also has a similar description.*

Proof. By Remark 4.4, the map $T_X = \mathsf{R}(M_X) \to \mathbb{Q} = \mathsf{R}(M_{\operatorname{Spec} k})$ induced by $M_X \to M_{\operatorname{Spec} k} = \mathbf{1}_k$ can be viewed as the map $T_X \to \mathbb{Q}$ induced by $x \in X^t$. Remember that the opposite of the multiplicative realization functor $\mathrm{CAlg}(\mathsf{DM}^{\otimes}(k))^{op} \to \mathrm{CAlg}_{\mathbb{Q}}^{op} = \mathrm{Aff}_{\mathbb{Q}}$ preserves small limits. Therefore, the derived affine group scheme $G(X,x)$ is the Cech nerve of $\operatorname{Spec}\mathbb{Q} \to \operatorname{Spec} T_X$ in $\mathrm{Aff}_{\mathbb{Q}}$. The second claim is clear from this argument. □

5. Sullivan models and computational results

In rational homotopy theory, an inductive construction of a Sullivan model is quite powerful. Let S be a topological space and $A_{PL}(S)$ the commutative dg algebra of polynomial differential forms. As in Section 4 we write $A_{PL,\infty}(S)$ for the image of $A_{PL}(S)$ in $\mathrm{CAlg}_{\mathbb{Q}}$. Let $\mathbb{F}_{\mathbb{Q}}$ denote the free functor $\mathrm{Mod}_{\mathbb{Q}} \simeq \mathsf{D}(\mathbb{Q}) \to \mathrm{CAlg}_{\mathbb{Q}}$ which is defined to be a left adjoint to the forgetful functor $\mathrm{CAlg}_{\mathbb{Q}} \to \mathsf{D}(\mathbb{Q})$. Contrary to genuine commutative dg algebras, in the setting of $\mathrm{CAlg}_{\mathbb{Q}}$ it is nonsense to say what a underlying graded algebra is. But in the language of $\mathrm{CAlg}_{\mathbb{Q}}$, the inductive construction describes $A_{PL,\infty}(S)$ as a colimit of a sequence

$$\mathbb{Q} \simeq A_0 \to A_1 \to \cdots \to A_n \to A_{n+1} \to \cdots$$

such that for any $n \geq 0$, A_{n+1} fits in the pushout diagram of the form

$$\begin{array}{ccc} \mathbb{F}_{\mathbb{Q}}(V) & \longrightarrow & A_n \\ \downarrow & & \downarrow \\ \mathbb{Q} & \longrightarrow & A_{n+1} \end{array}$$

in $\mathrm{CAlg}_{\mathbb{Q}}$ where V is a $\mathbb{Z}$-graded vector space over $\mathbb{Q}$ regarded as an object in $\mathsf{D}(\mathbb{Q})$, and the vertical arrow is $\mathbb{F}_{\mathbb{Q}}(V) \to \mathbb{F}_{\mathbb{Q}}(0) \simeq \mathbb{Q}$ induced by $V \to 0$. Note that $\mathbb{F}_{\mathbb{Q}}(V) \to A_n$ is determined by a morphism $V \to A_n$ in $\mathsf{D}(\mathbb{Q})$. Suppose that V is concentrated in a fixed positive degree n, i.e., $V^i = 0$ for $i \neq n$, and the $\mathbb{Q}$-vector space V^n is finite dimensional. Then $\mathbb{F}_{\mathbb{Q}}(V)$ is the commutative dg algebra that corresponds to the rational homotopy type of the Eilenberg-MacLane space $K((V^n)^{\vee}, n)$. Informally, the above sequence may be thought of as a presentation of $A_{PL}(S)$ as a "successive extension" of "simple pieces" of the form $\mathbb{F}_{\mathbb{Q}}(V[1])$.

We will apply this approach to $\mathrm{CAlg}(\mathsf{DM}^{\otimes}(k))$ and study cohomological motivic algebras. Free commutative algebra objects in $\mathsf{DM}(k)$ play the role of free commutative dg algebras. Actually, from the Tannakian viewpoint, such free objects are quite "simple" objects, see Remark 7.11. Put another way, presentations of successive extensions by free objects is useful for computations of a motivic counterpart of rational homotopy groups. We will inroduce the notion of cotangent motives in Section 6. We then apply the study of structures of cohomological motivic algebras in this section to obtain explicit descriptions of cotangent motives (Theorem 6.13).

In this section, we work with rational coefficients, but $\mathbb{Q}$ can be replaced by any field K of characteristic zero.

5.1 We will study some "relatively elementary" examples such as projective spaces. We also hope that the reader will get the feeling of the idea of the constructions of "Sullivan models" of cohomological motivic algebras in $\mathrm{CAlg}(\mathsf{DM}^{\otimes}(k))$.

Recall free commutative algebras in a general setting.

Definition 5.1. Let $\mathcal{C}^{\otimes}$ be a symmetric monoidal ∞-category that has small colimits and the tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves small colimits separately in each variable. Let $u_{\mathcal{C}} : \mathrm{CAlg}(\mathcal{C}^{\otimes}) \to \mathcal{C}$ be the forgetful functor. By [33, 3.1.3], there exists a left adjoint

$$\mathbb{F}_{\mathcal{C}} : \mathcal{C} \longrightarrow \mathrm{CAlg}(\mathcal{C}^{\otimes})$$

to $u_{\mathcal{C}}$, which we shall call the free functor of $\mathcal{C}^{\otimes}$ ([33] treats a broader setting). Given $C \in \mathcal{C}$ we refer to $\mathbb{F}_{\mathcal{C}}(C)$ as the free commutative algebra (object) generated by C. We often omit the notation $u_{\mathcal{C}}$.

For $A \in \mathrm{CAlg}(\mathcal{C}^{\otimes})$, by the adjunction, a morphism $f : \mathbb{F}_{\mathcal{C}}(C) \to A$ corresponds to the composite $\alpha : C \stackrel{\mathrm{unit}}{\to} u_{\mathcal{C}}(\mathbb{F}_{\mathcal{C}}(C)) \stackrel{u_{\mathcal{C}}(f)}{\to} u_{\mathcal{C}}(A)$ in $\mathcal{C}$. We say that $f : \mathbb{F}_{\mathcal{C}}(C) \to A$ is classified by α.

According to [33, 3.1.3.13], the underlying object $\mathbb{F}_{\mathcal{C}}(C)$ is equivalent to the coproduct $\sqcup_{n\ge 0} \mathrm{Sym}^n(C)$ in $\mathcal{C}$, where $\mathrm{Sym}^n_{\mathcal{C}}(C)$ is the n-fold symmetric product (we usually omit the subscript when the setting is obvious). If $\mathcal{D}^{\otimes}$ is a symmetric monoidal ∞-category having the same property and $F : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ is a colimit-preserving functor, then there is a canonical equivalence $\mathbb{F}_{\mathcal{D}}(F(C)) \stackrel{\sim}{\to} F(\mathbb{F}_{\mathcal{C}}(C))$ for any $C \in \mathcal{C}$.

5.1.1 Contrary to $\mathrm{CAlg}_{\mathbb{Q}}$, explicit computations of pushouts in $\mathrm{CAlg}(\mathsf{DM}^{\otimes}(k))$ are very complicated. To achieve our explicit study, we introduce some new devises which we will use.

Let GL_d be the general linear algebraic group over $\mathbb{Q}$. Let $\mathrm{Vect}^{\otimes}(\mathrm{GL}_d)$ be the symmetric monoidal abelian category of (possibly infinite dimensional) representations of GL_d, that is, $\mathbb{Q}$-vector spaces with action of GL_d. The symmetric monoidal category $\mathrm{Comp}(\mathrm{GL}_d) := \mathrm{Comp}(\mathrm{Vect}(\mathrm{GL}_d))$ of (possibly unbounded) cochain complexes admits a proper combinatorial symmetric monoidal model structure such that (i) $f : C \to C'$ is a weak equivalence if a quasi-isomorphism, (ii) every object is cofibrant, and (iii) $\{\iota_M : S^{n+1}M \hookrightarrow D^nM\}_{M\in I, n\in\mathbb{Z}}$ is a set of generating cofibrations consisting of natural inclusions, where I is the set of irreducible representations of GL_d, and S^nM (reps. D^nM) in $\mathrm{Comp}(\mathrm{GL}_d)$ defined by $(S^nM)^n = M$ and $(S^nM)^m = 0$ for $m \neq n$ (resp. $(D^nM)^n = (D^nM)^{n+1} = M$, $D^mM = 0$ for $m \neq n, n+1$, and $d : (D^nM)^n \to (D^nM)^{n+1}$ is the identity), see [26, Section 2.3], [11, Corollary 3.5] for details. Let $\mathrm{Rep}^{\otimes}(\mathrm{GL}_d)$ be the symmetric monoidal ∞-category, which is obtained from $\mathrm{Comp}(\mathrm{GL}_d)$ by inverting quasi-isomorphisms. Let $\mathrm{CAlg}(\mathrm{Rep}^{\otimes}(\mathrm{GL}_d))$ be the ∞-category of commutative algebra objects in $\mathrm{Rep}^{\otimes}(\mathrm{GL}_d)$.

Proposition 5.2. *Let M be an object of $\mathsf{DM}(k)$. Suppose that $(d{+}1)$-fold wedge product $\wedge^{n+1}(M)$ is 0, and $\wedge^d(M) \neq 0$. Then there exists a symmetric monoidal colimit-preserving functor*

$$\mathrm{Rep}^{\otimes}(\mathrm{GL}_d) \to \mathsf{DM}^{\otimes}(M)$$

which sends the standard representation of GL_d to M. Moreover, such a functor is unique up to a contractible space of choices.

Proof. This is a consequence of [26, Theorem 3.1, Proposition 6.1]. ∎

Lemma 5.3. *Let* $\mathrm{CAlg}(\mathrm{Comp}(\mathrm{GL}_d))$ *denote the category of commutative algebra objects in* $\mathrm{Comp}(\mathrm{GL}_d)$*. (We may think of an object as a commutative dg algebra equipped with action of* GL_d*.) Then there is a combinatorial model structure on* $\mathrm{CAlg}(\mathrm{Comp}(\mathrm{GL}_d))$ *where a map* $f : A \to A'$ *in* $\mathrm{CAlg}(\mathrm{Comp}(\mathrm{GL}_d))$ *is a weak equivalence (resp. a fibration) if* f *is a weak equivalence (reps. a fibration) in the underlying category* $\mathrm{Comp}(\mathrm{GL}_d)$*. In addition, if* $\mathrm{CAlg}(\mathrm{Comp}(\mathrm{GL}_d))[W^{-1}]$ *denotes the* ∞*-category obtained from the full subcategory of cofibrants in* $\mathrm{CAlg}(\mathrm{Comp}(\mathrm{GL}_d))$ *by inverting weak equivalences, then the canonical functor*

$$\mathrm{CAlg}(\mathrm{Comp}(\mathrm{GL}_d))[W^{-1}] \to \mathrm{CAlg}(\mathrm{Rep}^{\otimes}(\mathrm{GL}_d))$$

is an equivalence of ∞*-categories.*

Proof. Thanks to [33, 4.5.4.4, 4.5.4.6, 4.5.4.7], it is enough to prove that every cofibration in $\mathrm{Comp}(\mathrm{GL}_d)$ is a power cofibration in the sense of [33, 4.5.4.2]. To this end, we first observe that a morphism $f : C \to C'$ in $\mathrm{Comp}(G) := \mathrm{Comp}(\mathrm{Vect}(G))$ is a cofibration if and only if f is a monomorphism when G is either GL_d or a symmetric group Σ_n. Let M be an irreducible representation of G. By the representation theory of GL_d or Σ_n, $\mathrm{Vect}(G)$ is semi-simple and $\mathrm{Hom}_{\mathrm{Vect}(G)}(M, M) = \mathbb{Q}$ for any irreducible representation M of G. Let $\xi_M : \mathrm{Vect}(G) \to \mathrm{Vect}$ be the functor to the category of $\mathbb{Q}$-vector spaces, that is given by $N \mapsto \mathrm{Hom}_{\mathrm{Vect}(G)}(M, N)$. Taking the product indexed by the set $I(G)$ of irreducible representations of G, we have $\sqcap_{M\in I(G)}\xi_M : \mathrm{Vect}(G) \to \sqcap_{I(G)} \mathrm{Vect}$. Note that this functor is an equivalence of categories and induces an equivalence $\mathrm{Comp}(G) \to \sqcap_{I(G)} \mathrm{Comp}(\mathbb{Q})$ in the obvious way. For an irreducible representation P, $S^{n+1}P \to D^nP$ corresponds to a morphism $\{f_M\}_{M\in I(G)}$ in $\sqcap_{I(G)} \mathrm{Comp}(\mathbb{Q})$ such that $f_P : S^{n+1}\mathbb{Q} \to D^n\mathbb{Q}$ and $f_M = 0$ if $M \neq P$ through this equivalence. Therefore, it will suffice to show that the smallest weakly saturated class containing $\{S^{n+1}\mathbb{Q} \to D^n\mathbb{Q}\}_{n\in\mathbb{Z}}$ coincides with a collection of monomorphisms in $\mathrm{Comp}(\mathbb{Q})$. In fact, $\{S^{n+1}\mathbb{Q} \to D^n\mathbb{Q}\}_{n\in\mathbb{Z}}$ is a set of generating cofibrations in the projective model structure of $\mathrm{Comp}(\mathbb{Q})$. Since $\mathbb{Q}$ is a field, a morphism in $\mathrm{Comp}(\mathbb{Q})$ is a cofibration with respect to the projective model structure exactly when it is a monomorphism. Thus, we conclude that a morphism $f : C \to C'$ in $\mathrm{Comp}(G)$ is a cofibration if and only if f is a monomorphism.

Next we prove that a cofibration $f : C \to C'$ of $\mathrm{Comp}(\mathrm{GL}_d)$ is a power cofibration. We say that f is a power cofibration if a Σ_n-equivariant map $\wedge^n(f) : \square^n(f) \to (C')^{\otimes n}$ is a cofibration in $\mathrm{Comp}(\mathrm{GL}_d)^{\Sigma_n}$ for any $n \geq 0$. Here $\mathrm{Comp}(\mathrm{GL}_d)^{\Sigma_n}$ is the category of objects in $\mathrm{Comp}(\mathrm{GL}_d)$ endowed with action of the symmetric group Σ_n, which is equipped with the projective model structure. We refer to [33, 4.5.4.1] for these definitions and notations. Let $U : \mathrm{Comp}(\mathrm{GL}_d) \to \mathrm{Comp}(\mathbb{Q})$ be the forgetful functor, that is a symmetric monoidal left adjoint. It follows that $\wedge^n U(f) \simeq U(\wedge^n(f))$. Suppose that f is a cofibration. Then $U(f)$ is a cofibration with respect to the projective model structure because it is a monomorphism. According to [33, 7.1.4.7], $U(f)$ is a power cofibration. Thus by the above consideration $U(\wedge^n(f)) \simeq \wedge^n U(f)$ is a monomorphism. Then $\wedge^n(f)$ is a monomorphism in $\mathrm{Comp}(\mathrm{GL}_d)$. Note that there is a canonical equivalence $(\sqcap_I \mathrm{Comp}(\mathbb{Q}))^{\Sigma_n} \xrightarrow{\sim} \sqcap_I(\mathrm{Comp}(\mathbb{Q})^{\Sigma_n}) = \sqcap_I \mathrm{Comp}(\Sigma_n)$. The image of $\wedge^n(f)$ in $\sqcap_I(\mathrm{Comp}(\mathbb{Q})^{\Sigma_n})$ is a monomorphism. Again by the above consideration, the image is a cofibration in $\sqcap_I(\mathrm{Comp}(\mathbb{Q})^{\Sigma_n})$ endowed with the projective structure. Therefore, $\wedge^n(f)$ has the left lifting property with respect to epimorphic quasi-isomorphisms in $\mathrm{Comp}(\mathrm{GL}_d)^{\Sigma_n}$, namely, it is a cofibration. □

Let $u : \mathrm{CAlg}(\mathrm{Comp}(\mathrm{GL}_d)) \to \mathrm{Comp}(\mathrm{GL}_d)$ be the forgetful functor. By the definition of the model structure on $\mathrm{CAlg}(\mathrm{Comp}(\mathrm{GL}_d))$ in Lemma 5.3, it is a right Quillen functor. We denote by

$\mathbb{F}_{\mathrm{Comp}(\mathrm{GL}_d)} : \mathrm{Comp}(\mathrm{GL}_d) \to \mathrm{CAlg}(\mathrm{Comp}(\mathrm{GL}_d))$ a left Quillen functor to u. It is the free algebra functor of $\mathrm{Comp}(\mathrm{GL}_d)$. Since every object in $\mathrm{Comp}(\mathrm{GL}_d)$ is cofibrant, thus $\mathbb{F}_{\mathrm{Comp}(\mathrm{GL}_d)}$ preserves weak equivalences; that is to say, it is "derived". Let $u_\infty : \mathrm{CAlg}(\mathrm{Rep}(\mathrm{GL}_d)) \to \mathrm{Rep}(\mathrm{GL}_d)$ be the forgetful functor of ∞-categories. We write $\mathbb{F}_{\mathrm{Rep}(\mathrm{GL}_d)} : \mathrm{Rep}(\mathrm{GL}_d) \to \mathrm{CAlg}(\mathrm{Rep}(\mathrm{GL}_d))$ for the free algebra functor of $\mathrm{Rep}^{\otimes}(\mathrm{GL}_d)$. The following Lemma guarantees compatibility between $\mathbb{F}_{\mathrm{Rep}(\mathrm{GL}_d)}$ and $\mathbb{F}_{\mathrm{Comp}(\mathrm{GL}_d)}$.

Lemma 5.4. *Let C be an object in $\mathrm{Comp}(\mathrm{GL}_d)$. By abuse of notation, we write C (resp. $\mathbb{F}_{\mathrm{Comp}(\mathrm{GL}_d)}(C)$) for the images of the cofibrant object C (resp. $\mathbb{F}_{\mathrm{Comp}(\mathrm{GL}_d)}(C)$) in $\mathrm{Rep}(\mathrm{GL}_d)$ (resp. $\mathrm{CAlg}(\mathrm{Rep}(\mathrm{GL}_d))$). Then there is a canonical equivalence $\mathbb{F}_{\mathrm{Comp}(\mathrm{GL}_d)}(C) \simeq \mathbb{F}_{\mathrm{Rep}(\mathrm{GL}_d)}(C)$ in $\mathrm{CAlg}(\mathrm{Rep}(\mathrm{GL}_d))$, which commutes with $C \to u_\infty(\mathbb{F}_{\mathrm{Comp}(\mathrm{GL}_d)}(C))$ and $C \to u_\infty(\mathbb{F}_{\mathrm{Rep}(\mathrm{GL}_d)}(C))$.*

Proof. The forgetful functors u and u_∞ commute with canonical maps $\mathrm{CAlg}(\mathrm{Comp}(\mathrm{GL}_d)) \to \mathrm{CAlg}(\mathrm{Rep}(\mathrm{GL}_d))$ and $\mathrm{Comp}(\mathrm{GL}_d) \to \mathrm{Rep}(\mathrm{GL}_d)$. By Lemma 5.3 we identify the induced functor $\mathrm{h}(u_\infty) : \mathrm{h}(\mathrm{CAlg}(\mathrm{Rep}(\mathrm{GL}_d))) \to \mathrm{h}(\mathrm{Rep}(\mathrm{GL}_d))$ of homotopy categories with the right adjoint

$$\overline{u} : \mathrm{h}(\mathrm{CAlg}(\mathrm{Comp}(\mathrm{GL}_d))[W^{-1}]) \to \mathrm{h}(\mathrm{Rep}(\mathrm{GL}_d))$$

of homotopy categories induced by the right Quillen functor u. Thus, we can identify the left adjoint $\mathrm{h}(\mathbb{F}_{\mathrm{Rep}(\mathrm{GL}_d)}) : \mathrm{h}(\mathrm{Rep}(\mathrm{GL}_d)) \to \mathrm{h}(\mathrm{CAlg}(\mathrm{Rep}(\mathrm{GL}_d)))$ with $\mathrm{h}(\mathbb{F}_{\mathrm{Comp}(\mathrm{GL}_d)}) : \mathrm{h}(\mathrm{Rep}(\mathrm{GL}_d)) \to \mathrm{h}(\mathrm{CAlg}(\mathrm{Comp}(\mathrm{GL}_d))[W^{-1}])$ induced by $\mathbb{F}_{\mathrm{Comp}(\mathrm{GL}_d)}$. □

Proposition 5.5. *Let A be a cofibrant object in $\mathrm{CAlg}(\mathrm{Comp}(\mathrm{GL}_d))$ and let $\alpha : C \to u(A)$ be a morphism in $\mathrm{Comp}(\mathrm{GL}_d)$. Let $\phi_\alpha : \mathbb{F}_{\mathrm{Comp}(\mathrm{GL}_d)}(C) \to A$ be the morphism classified by α. Let $\iota : S^0\mathbb{Q} \hookrightarrow D^{-1}\mathbb{Q}$ be the cofibration in $\mathrm{Comp}(\mathrm{GL}_d)$, where $\mathbb{Q}$ here denotes the unit object in $\mathrm{Comp}(\mathrm{GL}_d)$ (we abuse notation). Let $\mathbb{F}_{\mathrm{Comp}(\mathrm{GL}_d)}(C) \to \mathbb{F}_{\mathrm{Comp}(\mathrm{GL}_d)}(C \otimes (D^{-1}\mathbb{Q}))$ be the morphism induced by $C \otimes \iota : C \simeq C \otimes (S^0\mathbb{Q}) \to C \otimes (D^{-1}\mathbb{Q})$. Let $A\langle\alpha\rangle$ be the pushout of the following diagram in $\mathrm{CAlg}(\mathrm{Comp}(\mathrm{GL}_d))$:*

$$\begin{array}{ccc} \mathbb{F}_{\mathrm{Comp}(\mathrm{GL}_d)}(C) & \xrightarrow{\phi_\alpha} & A \\ \downarrow & & \downarrow \\ \mathbb{F}_{\mathrm{Comp}(\mathrm{GL}_d)}(C \otimes (D^{-1}\mathbb{Q})) & \longrightarrow & A\langle\alpha\rangle. \end{array}$$

Then this diagram is a homotopy pushout. See Remark 5.6 for the explicit presentation of $A\langle\alpha\rangle$.

Remark 5.6. The commutative algebra object $A\langle\alpha\rangle$ is regarded as a commutative dg algebra endowed with an action of GL_d. The explicit presentation of $A\langle\alpha\rangle$ is described as follows (see the proof of Proposition 5.5). For simplicity, we suppose that differential of C is zero and we view it as a graded vector space with an action of GL_d. This assumption is not essential in practice because $\mathrm{Vect}(\mathrm{GL}_d)$ is semi-simple. Let $\overline{A}$ be the underlying graded algebra of A obtained by forgetting the differential. The underlying graded algebra of $A\langle\alpha\rangle$ is given by the tensor product $\overline{A} \otimes \mathbb{F}_{\mathrm{Comp}(\mathrm{GL}_d)}(C[1])$ of commutative graded algebras with the action of GL_d. If one forgets the action of GL_d on $\mathbb{F}_{\mathrm{Comp}(\mathrm{GL}_d)}(C[1])$, then it is the free commutative graded algebra generated by the underlying graded algebra of $C[1]$. The differential on $\overline{A} \otimes \mathbb{F}_{\mathrm{Comp}(\mathrm{GL}_d)}(C[1])$ is given by the differential on A and $\partial|_C = \alpha$. When GL_d is the trivial, i.e., $d = 0$ or one forgets the action of GL_d, then the construction of $A\langle\alpha\rangle$ is classical, see [22, 2.2.2].

Example 5.7. Let $\mathbb{G}_m = \mathrm{GL}_1$. Let χ_i in $\mathrm{Comp}(\mathbb{G}_m)$ be one dimensional representation of $\mathbb{G}_m$ of weight i placed in degree zero. That is, χ_i can be regarded as the representation $\mathbb{G}_m \to \mathbb{G}_m = \mathrm{GL}_1$ associated to $\mathbb{Q}[t, t^{-1}] \to \mathbb{Q}[t, t^{-1}]$ given by $t \mapsto t^i$ (see e.g. [8, 5.2]). Let $A = \mathbb{F}_{\mathrm{Comp}(\mathbb{G}_m)}(\chi_1[-2])$ be the free commutative algebra generated by $\chi_1[-2]$. The underlying cochain complex is $\oplus_{i\geq 0}\chi_i[-2i]$ with zero differential. Let $\alpha : \chi_{n+1}[-2n-2] \to \oplus_{i\geq 0}\chi_i[-2i] = A$ be the canonical inclusion. Let us consider $A\langle\alpha\rangle$. Note that $\mathbb{F}_{\mathrm{Comp}(\mathbb{G}_m)}(\chi_{n+1}[-2n-1])$ is the trivial square zero extension $\chi_0 \oplus \chi_{n+1}[-2n-1]$ by $\chi_{n+1}[-2n-1]$ (since the generator is in the odd degree). The underlying graded algebra is $(\oplus_{i\geq 0}\chi_i[-2i]) \otimes (\chi_0 \oplus \chi_{n+1}[-2n-1])$. The non-zero part of differential is given by "identities" $\chi_i[-2i] \otimes \chi_{n+1}[-2n-1] \to \chi_{i+n+1}[-2i-2n-2] \otimes \chi_0$ for $i \geq 0$.

The standard consequence of Proposition 5.5 is

Corollary 5.8. *The image of the square diagram in Proposition 5.5 in* $\mathrm{CAlg}(\mathrm{Rep}(\mathrm{GL}_d))$ *is a pushout diagram. We remark that the image of* $\mathbb{F}_{\mathrm{Comp}(\mathrm{GL}_d)}(C)$ *and* $\mathbb{F}_{\mathrm{Comp}(\mathrm{GL}_d)}(C \otimes (D^{-1}K))$ *in* $\mathrm{CAlg}(\mathrm{Rep}(\mathrm{GL}_d))$ *are equivalent to* $\mathbb{F}_{\mathrm{Rep}(\mathrm{GL}_d)}(C)$ *and the unit algebra, respectively (Lemma 5.4).*

Proof of Proposition 5.5. Let B be a pushout of $C \otimes D^{-1}\mathbb{Q} \leftarrow C \to u(A)$ in $\mathrm{Comp}(\mathrm{GL}_d)$, that is the standard mapping cone $(u(A) \oplus C[1], d)$ of $\alpha : C \to u(A)$. Since $u(A)$ is cofibrant and $C \otimes S^0K \to C \otimes D^{-1}K$ is a cofibration, B is a homotopy pushout, see e.g. [32, A.2.4.4]. Then we have the commutative diagram

$$\begin{array}{ccccc}
\mathbb{F}_{\mathrm{Comp}(\mathrm{GL}_d)}(C) & \longrightarrow & \mathbb{F}_{\mathrm{Comp}(\mathrm{GL}_d)}(u(A)) & \longrightarrow & A \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{F}_{\mathrm{Comp}(\mathrm{GL}_d)}(C \otimes D^{-1}K) & \longrightarrow & \mathbb{F}_{\mathrm{Comp}(\mathrm{GL}_d)}(B) & \longrightarrow & A\langle\alpha\rangle
\end{array}$$

that consists of pushout squares. The upper right horizontal map is the counit map. Since A is cofibrant and the left vertical arrow is a cofibration, again by [32, A.2.4.4] both left and right (and the outer) squares are homotopy pushouts, as claimed. The explicit structure of $A\langle\alpha\rangle$ in Remark 5.6 can easily be seen from the right pushout. □

5.1.2 We will consider the n-dimensional projective space $\mathbb{P}^n$ over a perfect field k.

We denote by $\mathbb{F}_{\mathsf{DM}(k)} : \mathsf{DM}(k) \to \mathrm{CAlg}(\mathsf{DM}^{\otimes}(k))$ the free algebra functor of $\mathsf{DM}^{\otimes}(k)$. For ease of notation, we put $\mathbb{F} := \mathbb{F}_{\mathsf{DM}(k)}$.

By the projective bundle theorem, there is a decomposition

$$M_{\mathbb{P}^n} \simeq M(\mathbb{P}^n)^{\vee} \simeq \mathbf{1}_k \oplus \mathbf{1}_k(-1)[-2] \oplus \ldots \oplus \mathbf{1}_k(-n)[-2n] = \oplus_{i=0}^{n}\mathbf{1}_k(-i)[-2i]$$

in $\mathsf{DM}(k)$, see e.g. [37, Lec.15]. Consider the inclusion $\iota : \mathbf{1}_k(-1)[-2] \hookrightarrow M_{\mathbb{P}^n} \simeq \oplus_{i=0}^{n}\mathbf{1}_k(-i)[-2i]$ that is a morphism in $\mathsf{DM}(k)$. It gives rise to a morphism

$$f : \mathbb{F}(\mathbf{1}_k(-1)[-2]) \to M_{\mathbb{P}^n}$$

in $\mathrm{CAlg}(\mathsf{DM}^{\otimes}(k))$, that is classified by ι. We note that $\mathbb{F}(\mathbf{1}_k(-1)[-2]) \simeq \oplus_{i\geq 0}\mathbf{1}(-i)[-2i]$ in $\mathsf{DM}(k)$. Observe that for $j > n$, the composite

$$\mathbf{1}_k(-j)[-2j] \hookrightarrow \oplus_{i\geq 0}\mathbf{1}(-i)[-2i] \simeq \mathbb{F}(\mathbf{1}_k(-1)[-2]) \to M_{\mathbb{P}^n}$$

is null homotopic. Indeed, the morphism $\mathbf{1}_k(-j)[-2j] \to \mathbf{1}_k(-i)[-2i]$ is null homotopic for $0 \le i \le n$ since $\mathbf{1}_k(j)[2j] \otimes (\mathbf{1}_k(-j)[-2j] \to \mathbf{1}_k(-i)[-2i])$ corresponds to an element of motivic cohomology $H_M^{2j-2i}(\operatorname{Spec} k, j-i) \simeq \mathrm{CH}^{j-i}(\operatorname{Spec} k) = 0$. Here $\mathrm{CH}^p(-)$ denotes the p-th Chow group, and the comparison isomorphism between motivic cohomology and (higher) Chow groups is due to Voevodsky. Next we let

$$g : \mathbb{F}(\mathbf{1}_k(-n-1)[-2n-2]) \to \mathbb{F}(\mathbf{1}_k(-1)[-2])$$

be a morphism that is classified by the inclusion $\mathbf{1}_k(-n-1)[-2n-2] \hookrightarrow \mathbb{F}(\mathbf{1}_k(-1)[-2])$. Consider the morphism $h : \mathbb{F}(\mathbf{1}_k(-n-1)[-2n-2]) \to \mathbb{F}(0) \simeq \mathbf{1}_k$ induced by $\mathbf{1}_k(-n-1)[-2n-2] \to 0$. Take a pushout

$$S_{\mathbb{P}^n} := \mathbb{F}_{\mathsf{DM}(k)}(\mathbf{1}_k(-1)[-2]) \otimes_{\mathbb{F}_{\mathsf{DM}(k)}(\mathbf{1}_k(-n-1)[-2n-2])} \mathbf{1}_k$$

along h in $\mathrm{CAlg}(\mathsf{DM}^{\otimes}(k))$. Note that $f \circ g$ factors through $h : \mathbb{F}(\mathbf{1}_k(-n-1)[-2n-2]) \to \mathbb{F}(0) \simeq \mathbf{1}_k$ because $\mathbf{1}_k(-n-1)[-2n-2] \to M_{\mathbb{P}^n}$ is a zero map in the homotopy category $\mathrm{h}(\mathrm{CAlg}(\mathsf{DM}(k)))$. Consequently, by the universal property of the pushout we obtain the induced morphism

$$S_{\mathbb{P}^n} \to M_{\mathbb{P}^n}.$$

Proposition 5.9. *The morphism $S_{\mathbb{P}^n} \to M_{\mathbb{P}^n}$ is an equivalence in* $\mathrm{CAlg}(\mathsf{DM}^{\otimes}(k))$.

Proof. We first claim that $\oplus_{i\ge 0}\mathbf{1}_k(-i)[-2i] \simeq \mathbb{F}(\mathbf{1}_k(-1)[-2]) \to M_{\mathbb{P}^n} \simeq \oplus_{i=0}^n \mathbf{1}_k(-i)[-2i]$ induces an equivalence $\mathbb{F}(\mathbf{1}_k(-1)[-2]) \supset \mathbf{1}_i(-i)[-2i] \xrightarrow{\sim} \mathbf{1}_i(-i)[-2i] \subset M_{\mathbb{P}^n}$ for $0 \le i \le n$. As discussed before this Proposition, $\mathbf{1}_i(-i)[-2i] \subset \mathbb{F}(\mathbf{1}_k(-1)[-2]) \to M_{\mathbb{P}^n}$ is null homotopic if $i > n$ because $\mathrm{Hom}_{\mathrm{h}(\mathsf{DM}(k))}(\mathbf{1}_k(a)[2a], \mathbf{1}_k(b)[2b])$ is $\mathbb{Q}$ (resp. 0) if $a = b$ (resp. $a \ne b$). Consider the dual $M(\mathbb{P}^n) \simeq \oplus_{i=0}^n \mathbf{1}_k(i)[2i]$ of the isomorphism $M_{\mathbb{P}^n} \simeq \oplus_{i=0}^n \mathbf{1}_k(-i)[-2i]$. Recall that the Chow ring $\mathrm{CH}^*(\mathbb{P}^n)$ is isomorphic to $\mathbb{Z}[H]/(H^{n+1})$ where $H \in \mathrm{CH}^1(\mathbb{P}^n)$ is a class of a hyperplane. The projection $M(\mathbb{P}^n) \to \mathbf{1}_k(i)[2i]$ corresponds to a generator of Chow group $\mathbb{Q} = \mathrm{CH}^i(\mathbb{P}^n) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq H_M^{2i}(X, i) \simeq \mathrm{Hom}_{\mathrm{h}(\mathsf{DM}(k))}(M(\mathbb{P}^n), \mathbf{1}_k(i)[2i])$. Using scalar multiplication (if necessary), we may and will assume that $M(\mathbb{P}^n) \to \mathbf{1}_k(i)[2i]$ corresponds to H^i. Now we prove our claim by induction on i. By the construction, the case of $i = 1$ is clear. We suppose that the case $i(< n-1)$ is true. We will show the case $i+1$. By Lemma 5.19, $\mathbb{F}(\mathbf{1}_k(-1)[-2])$ in the homotopy category $\mathrm{h}(\mathsf{DM}(k))$ is also regarded as the free commutative algebra object lying in $\mathrm{CAlg}(\mathrm{h}(\mathsf{DM}^{\otimes}(k))$ generated by $\mathbf{1}_k(-1)[-2]$ in $\mathrm{h}(\mathsf{DM}^{\otimes}(k))$. Thus, the multiplication map $\mathbb{F}(\mathbf{1}_k(-1)[-2]) \otimes \mathbb{F}(\mathbf{1}_k(-1)[-2]) \to \mathbb{F}(\mathbf{1}_k(-1)[-2])$ induces an isomorphism from the component $\mathbf{1}_k(-a)[-2a] \otimes \mathbf{1}_k(-b)[-2b]$ in the domain to $\mathbf{1}_k(-a-b)[-2a-2b]$ in the target. Therefore, by the induction hypothesis and the compatibility of multiplication maps, if the multiplication $M_{\mathbb{P}^n} \otimes M_{\mathbb{P}^n} \to M_{\mathbb{P}^n}$ induces an isomorphism of the composite

$$\xi : \mathbf{1}_k(-1)[-2] \otimes \mathbf{1}_k(-i)[-2i] \hookrightarrow M_{\mathbb{P}^n} \otimes M_{\mathbb{P}^n} \to M_{\mathbb{P}^n} \to \mathbf{1}_k(-i-1)[-2i-2],$$

then $\mathbb{F}(\mathbf{1}_k(-1)[-2]) \to M_{\mathbb{P}^n}$ induces an isomorphism from the component $\mathbf{1}_k(-i-1)(-2i-2)$ in the domain to $\mathbf{1}_k(-i-1)[-2i-2] \subset M_{\mathbb{P}^n}$ (namely, the case $i+1$ holds). Note that the dual $M(\mathbb{P}^n) \to \mathbf{1}_k(i)[2i]$ of $\mathbf{1}_k(-i)[-2i] \to M_{\mathbb{P}^n}$ corresponds to the element $H^i \in \mathrm{CH}^i(\mathbb{P}^n)$ (for any i). Observe that the dual $M(\mathbb{P}^n) \to \mathbf{1}_k(i+1)[2i+2]$ of the composite $l : \mathbf{1}_k(-1)[-2] \otimes \mathbf{1}_k(-i)[-2i] \hookrightarrow M_{\mathbb{P}^n} \otimes M_{\mathbb{P}^n} \to M_{\mathbb{P}^n}$ corresponds to the intersection product $H^{i+1} = H \cdot H^i \in \mathrm{CH}(\mathbb{P}^n)$. To see this, recall that the product of motivic cohomology

$$\mathrm{Hom}_{\mathrm{h}(\mathsf{DM}(k))}(M(\mathbb{P}^n), \mathbf{1}_k(1)[2]) \otimes \mathrm{Hom}_{\mathrm{h}(\mathsf{DM}(k))}(M(\mathbb{P}^n), \mathbf{1}_k(i)[2i]) \\ \to \mathrm{Hom}_{\mathrm{h}(\mathsf{DM}(k))}(M(\mathbb{P}^n), \mathbf{1}_k(i+1)[2i+2])$$

is induced by the composition with $M(\mathbb{P}^n) \to M(\mathbb{P}^n) \otimes M(\mathbb{P}^n)$ defined by the diagonal map. By Lemma 5.22 below, the multiplication $M_{\mathbb{P}^n} \otimes M_{\mathbb{P}^n} \to M_{\mathbb{P}^n}$ is the dual of $M(\mathbb{P}^n) \to M(\mathbb{P}^n) \otimes M(\mathbb{P}^n)$. In addition, the product structure on motivic cohomology is compatible with that of (higher) Chow groups via the comparison isomorphism [28]. Therefore, we conclude that the dual of l corresponds to $H^{i+1} \in \mathrm{CH}^{i+1}(\mathbb{P}^n)$. It follows that ξ is an isomorphism.

Next, by Proposition 5.2 there is a symmetric monoidal colimit-preserving functor $F : \mathrm{Rep}^{\otimes}(\mathbb{G}_m) \to \mathsf{DM}^{\otimes}(k)$ which sends one dimensional representation χ_1 of weight one placed in degree zero to $\mathbf{1}_k(1)$. Put $\mathbb{G}_m := \mathrm{GL}_1$ and denote by χ_p one dimensional representation of weight p. Let $\mathbb{F}_{\mathrm{Comp}(\mathbb{G}_m)}(\chi_{-1}[-2])$ and $\mathbb{F}_{\mathrm{Comp}(\mathbb{G}_m)}(\chi_{-n-1}[-2n-2])$ be the free commutative algebra in $\mathrm{Comp}(\mathbb{G}_m)$ generated by $\chi_{-1}[-2]$ and $\chi_{-n-1}[-2n-2]$, respectively. Let $\mathbb{F}_{\mathrm{Comp}(\mathbb{G}_m)}(\chi_{-n-1}[-2n-2]) \to \mathbb{F}_{\mathrm{Comp}(\mathbb{G}_m)}(\chi_{-1}[-2])$ be the morphism classified by the inclusion $\alpha : \chi_{-n-1}[-2n-2] \hookrightarrow \mathbb{F}_{\mathrm{Comp}(\mathbb{G}_m)}(\chi_{-1}[-2])$. Take a homotopy pushout $\mathbb{F}_{\mathrm{Comp}(\mathbb{G}_m)}(\chi_{-1}[-2])\langle\alpha\rangle$, see Proposition 5.5. By Proposition 5.5 and Remark 5.6, an easy computation shows that $\mathbb{F}_{\mathrm{Comp}(\mathbb{G}_m)}(\chi_{-1}[-2])\langle\alpha\rangle \simeq \oplus_{i=0}^n \chi_{-i}[-2i]$ in $\mathrm{h}(\mathrm{Rep}(\mathbb{G}_m))$ and the natural map

$$\mathbb{F}_{\mathrm{Comp}(\mathbb{G}_m)}(\chi_{-1}[-2]) \simeq \oplus_{i\geq 0}\chi_{-i}[-2i] \to \mathbb{F}_{\mathrm{Comp}(\mathbb{G}_m)}(\chi_{-1}[-2])\langle\alpha\rangle \simeq \oplus_{i=0}^n \chi_{-i}[-2i]$$

is the projection (cf. Example 5.7). By abuse of notation, we will write χ_i, $\mathbb{F}_{\mathrm{Comp}(\mathbb{G}_m)}(\chi_{-1}[-2])$ and likewise for their images in $\mathrm{Rep}(\mathbb{G}_m)$ or $\mathrm{CAlg}(\mathrm{Rep}(\mathbb{G}_m))$. Note that F sends the χ_i to $\mathbf{1}_k(i)$ in $\mathsf{DM}(k)$. The left adjoint functor $\mathrm{CAlg}(F) : \mathrm{CAlg}(\mathrm{Rep}(\mathbb{G}_m)) \to \mathrm{CAlg}(\mathsf{DM}^{\otimes}(k))$ sends $\mathbb{F}_{\mathrm{Comp}(\mathbb{G}_m)}(\chi_{-n-1}[-2n-2]) \to \mathbb{F}_{\mathrm{Comp}(\mathbb{G}_m)}(\chi_{-1}[-2])$ to g. Then since $\mathrm{CAlg}(F)$ preserves pushouts, $\mathbb{F}_{\mathrm{Comp}(\mathbb{G}_m)}(\chi_{-1}[-2]) \to \mathbb{F}_{\mathrm{Comp}(\mathbb{G}_m)}(\chi_{-1}[-2])\langle\alpha\rangle$ maps to the canonical morphism $\mathbb{F}(\mathbf{1}_k(-1)[-2]) \to S_{\mathbb{P}^n}$. We see that the composite

$$\oplus_{i=0}^n \mathbf{1}_k(-i)[-2i] \hookrightarrow \oplus_{i\geq 0}\mathbf{1}_k(-i)[-2i] \simeq \mathbb{F}(\mathbf{1}_k(-1)[-2]) \to S_{\mathbb{P}^n} \simeq \oplus_{i=0}^n \mathbf{1}_k(-i)[-2i]$$

is an equivalence. Taking account of the first claim of this proof, we see that the underlying morphism $S_{\mathbb{P}^n} \to M_{\mathbb{P}^n}$ in $\mathsf{DM}(k)$ is an equivalence. Thus, $S_{\mathbb{P}^n} \to M_{\mathbb{P}^n}$ in $\mathsf{DM}(k)$ is an equivalence in $\mathrm{CAlg}(\mathsf{DM}^{\otimes}(k))$. □

Remark 5.10. Suppose that the base field k is embedded in $\mathbb{C}$. Let $\mathsf{R} : \mathrm{CAlg}(\mathsf{DM}^{\otimes}(k)) \to \mathrm{CAlg}_{\mathbb{Q}}$ be the multiplicative realization functor considered in Section 4. The multiplicative realization functor commutes with free algebra functors and the formation of colimits. Then the above construction of $S_{\mathbb{P}^n}$ and the equivalence $S_{\mathbb{P}^n} \simeq M_{\mathbb{P}^n}$ is compatible with the classical construction of a Sullivan model of $A_{PL}(\mathbb{CP}^n)$ where $\mathbb{CP}^n$ is the complex projective space. The morphism $\mathsf{R}(\mathbb{F}(\mathbf{1}_k(-1)[-2])) \simeq \mathbb{F}_{\mathbb{Q}}(\mathbb{Q}[-2]) \to \mathsf{R}(M_{\mathbb{P}^n}) \simeq A_{PL,\infty}(\mathbb{CP}^n)$ induced by f is given by a morphism $\mathbb{Q}[-2] \to A_{PL,\infty}(\mathbb{CP}^n)$ defined by a generator of $H^2(\mathbb{CP}^n, \mathbb{Q}) = \mathbb{Q}$. This is the first step of the construction of a Sullivan model. The subsequent steps are also compatible. See e.g. [21]. Also, we remark that $\pi_i(\mathbb{CP}^n) \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}$ if $i = 2, 2n+1$, and $\pi_i(\mathbb{CP}^n) \otimes_{\mathbb{Z}} \mathbb{Q} = 0$ if otherwise. See also Theorem 6.13 and Remark 6.14.

Remark 5.11. The object $M_{\mathbb{P}^n}$ lies in the full subcategory of mixed Tate motives in $\mathsf{DM}(k)$. But the above argument works for arbitrary perfect base fields and does not need a (conjectural) motivic t-structure.

5.1.3 Let $\mathbb{A}^n$ denote the n-dimensional affine space over a perfect field k. Let $X = \mathbb{A}^n - \{p\}$ be the open subscheme of $\mathbb{A}^n$ that is obtained by removing a k-rational point p. Let $j : X \to \mathbb{A}^n$

be the open immersion. By the dual of the Gysin triangle [37, 14.5], we have a distinguished triangle

$$\mathbf{1}_k(-n)[-2n] \to M_{\mathbb{A}^n} \xrightarrow{j^*} M_X$$

in the triangulated category $\mathrm{h}(\mathsf{DM}(k))$. Note that $M_{\mathbb{A}^n} \simeq \mathbf{1}_k$ and $\mathbf{1}_k(-n)[-2n] \to M_{\mathbb{A}^n}$ is null homotopic (see the case in 5.1.2). Hence we have an equivalence $M_X \simeq \mathbf{1}_k \oplus \mathbf{1}_k(-n)[-2n+1]$ in $\mathsf{DM}(k)$. We let $\mathbb{F}(\mathbf{1}_k(-n)[-2n+1]) \to M_X$ be a morphism in $\mathrm{CAlg}(\mathsf{DM}^{\otimes}(k))$, that is classified by the inclusion $\mathbf{1}_k(-n)[-2n+1] \hookrightarrow \mathbf{1}_k \oplus \mathbf{1}_k(-n)[-2n+1] \simeq M_X$.

Proposition 5.12. *The morphism $\mathbb{F}_{\mathsf{DM}(k)}(\mathbf{1}_k(-n)[-2n+1]) \to M_X$ is an equivalence.*

Proof. We continue to use the notation in the proof of Proposition 5.9 and the colimit-preserving symmetric monoidal functor $F : \mathrm{Rep}^{\otimes}(\mathbb{G}_m) \to \mathsf{DM}^{\otimes}(k)$. Let $\mathbb{F}_{\mathrm{Comp}(\mathbb{G}_m)}(\chi_{-n}[-2n+1])$ be the free algebra that belongs to $\mathrm{CAlg}(\mathrm{Comp}(\mathbb{G}_m))$ (keep in mind that it can be viewed as a commutative dg algebra endowed with an action of $\mathbb{G}_m$). Since the generator is concentrated in the odd degree $2n-1$, by the Koszul sign rule there is an isomorphism $\mathbb{F}_{\mathrm{Comp}(\mathbb{G}_m)}(\chi_{-n}[-2n+1]) \simeq \chi_0 \oplus \chi_{-n}[-2n+1]$ as objects in $\mathrm{Comp}(\mathbb{G}_m)$. The functor F carries $\mathbb{F}_{\mathrm{Comp}(\mathbb{G}_m)}(\chi_{-n}[-2n+1])$ to $\mathbb{F}(\mathbf{1}_k(-n)[-2n+1])$ in $\mathrm{CAlg}(\mathsf{DM}^{\otimes}(k))$. Thus, the underlying object of $\mathbb{F}(\mathbf{1}_k(-n)[-2n+1])$ is equivalent to $\mathbf{1}_k \oplus \mathbf{1}_k(-n)[-2n+1]$. Moreover, the canonical inclusion (unit map) $\mathbf{1}_k(-n)[-2n+1] \to \mathbb{F}(\mathbf{1}_k(-n)[-2n+1])$ is compatible with $\mathbf{1}_k(-n)[-2n+1] \hookrightarrow \mathbf{1}_k \oplus \mathbf{1}_k(-n)[-2n+1]$. Using these facts we deduce that $\mathbb{F}(\mathbf{1}_k(-n)[-2n+1]) \simeq \mathbf{1}_k \oplus \mathbf{1}_k(-n)[-2n+1] \to M_X \simeq \mathbf{1}_k \oplus \mathbf{1}_k(-n)[-2n+1]$ is an equivalence, as desired. □

Remark 5.13. Suppose that the base field k is embedded in $\mathbb{C}$. Then the complex manifold $X \times_{\mathrm{Spec}\,k} \mathrm{Spec}\,\mathbb{C}$ is homotopy equivalent to the $(2n-1)$-dimensional sphere S^{2n-1}. Proposition 5.12 is a motivic generalization of the fact that the free commutative dg algebra generated by one dimensional vector space placed in (cohomological) degree $2n-1$ is a Sullivan model of $A_{PL}(S^{2n-1})$ (cf. [16, Example 1 in page 142]).

5.1.4 Proposition 5.9 and 5.12 gives explicit "models" $S_{\mathbb{P}^n}$, $\mathbb{F}_{\mathsf{DM}(k)}(\mathbf{1}_k(-n)[-2n+1])$ of cohomological motivic algebras. The constructions of models have only finitely many steps. As in the classical rational homotopy theory, an inductive construction often consists of infinite steps. The following is such an example.

Let $Y = \mathbb{A}^n - \{p\} - \{q\}$ be the open subscheme of $\mathbb{A}^n$ that is obtained by removing two k-rational points p, q. Let $s : Y \to \mathrm{Spec}\,k$ denote the structure morphism.

Proposition 5.14. *Let $A_0 = \mathbf{1}_k$ be the unit algebra in $\mathrm{CAlg}(\mathsf{DM}^{\otimes}(k))$ and let $A_0 = \mathbf{1}_k \to M_Y$ be a unique morphism from the initial object $\mathbf{1}_k$ in $\mathrm{CAlg}(\mathsf{DM}^{\otimes}(k))$. Then there is a refinement of $A_0 \to M_Y$*

$$\mathbf{1}_k = A_0 \to A_1 \to A_2 \to \cdots \to A_i \to A_{i+1} \to \cdots \to M_Y$$

that satisfies the following properties:

(1) The canonical morphism $\varinjlim_{i\geq 0} A_i \to M_Y$ is an equivalence. Here $\varinjlim_i A_i$ be a colimit of the sequence in $\mathrm{CAlg}(\mathsf{DM}^{\otimes}(k))$.

(2) Let V_i be the kernel (homotopy fiber) of $A_i \to M_Y$ in $\mathsf{DM}(k)$ for any $i \geq 0$. Then for each $i \geq 0$, $A_i \to A_{i+1}$ is of the form $A_i \to A_i \otimes_{\mathbb{F}(V_i)} \mathbf{1}_k$ given by the pushout of $A_i \leftarrow \mathbb{F}(V_i) \to \mathbf{1}_k$ where $\mathbb{F}(V_i) \to A_i$ is classified by $V_i \to A_i$.

Moreover, for $n \geq 2$, one can explicitly compute each A_i in the sense explained below.

The first statement is a consequence of a more general fact, see Lemma 5.15 below. We explain the second statement, that is, the procedure of an explicit computation. We will compute the lower degrees A_1, A_2, A_3. We can apply the same procedure and arguments also to higher degrees and we leave it to the interested reader.

We continue to use the notation in Section 5.1.2, 5.1.3. As in the case of $X = \mathbb{A}^n - \{p\}$, applying the dual of Gysin triangle to the open immersion $Y \hookrightarrow \mathbb{A}^n$, we see that there is an equivalence $M_Y \simeq \mathbf{1}_k \oplus \mathbf{1}(-n)[-2n+1]^{\oplus 2}$ in $\mathsf{DM}(k)$. The morphism $s^* : \mathbf{1}_k \to \mathbf{1}_k \oplus \mathbf{1}(-n)[-2n+1]^{\oplus 2} \simeq M_Y$ induces an equivalence $\mathbf{1}_k \xrightarrow{\sim} \mathbf{1}_k \hookrightarrow \mathbf{1}_k \oplus \mathbf{1}_k(-n)[-2n+1]^{\oplus 2}$. Thus, $V_0 \simeq \mathbf{1}_k(-n)[-2n]^{\oplus 2}$. We then find that

$$A_1 = \mathbb{F}(0) \otimes_{\mathbb{F}(\mathbf{1}_k(-n)[-2n]^{\oplus 2})} \mathbb{F}(0) \simeq \mathbb{F}(0 \sqcup_{\mathbf{1}_k(-n)[-2n]^{\oplus 2}} 0) \simeq \mathbb{F}(\mathbf{1}_k(-n)[-2n+1]^{\oplus 2}).$$

The induced morphism $f : A_1 = \mathbb{F}(0) \otimes_{\mathbb{F}(\mathbf{1}_k(-n)[-2n]^{\oplus 2})} \mathbb{F}(0) \simeq \mathbb{F}(\mathbf{1}_k(-n)[-2n+1]^{\oplus 2}) \to M_Y$ is classified by the inclusion $\iota : \mathbf{1}_k(-n)[-2n+1]^{\oplus 2} \hookrightarrow \mathbf{1}_k \oplus \mathbf{1}_k(-n)[-2n+1]^{\oplus 2} \simeq M_Y$. Let $F : \mathrm{Rep}^{\otimes}(\mathbb{G}_m) \to \mathsf{DM}^{\otimes}(k)$ be the colimit-preserving symmetric monoidal functor which carries χ_1 to $\mathbf{1}_k(1)$ (cf. the proof of Proposition 5.9). Consider $\mathbb{F}_{\mathrm{Comp}(\mathbb{G}_m)}(\chi_{-n}[-2n+1]^{\oplus 2})$. The underlying object in $\mathrm{Comp}(\mathbb{G}_m)$ is isomorphic to $\mathbf{1}_k \oplus \chi_{-n}[-2n+1]^{\oplus 2} \oplus \mathrm{Sym}^2(\chi_{-n}[-2n+1]^{\oplus 2}) \simeq \mathbf{1}_k \oplus \chi_{-n}[-2n+1]^{\oplus 2} \oplus \chi_{-2n}[-4n+2]$. The image of $\mathbb{F}_{\mathrm{Comp}(\mathbb{G}_m)}(\chi_{-n}[-2n+1]^{\oplus 2})$ under $\mathrm{CAlg}(\mathrm{Rep}^{\otimes}(\mathbb{G}_m)) \to \mathrm{CAlg}(\mathsf{DM}^{\otimes}(k))$ is equivalent to A_1. The composite $\mathbf{1}_k \oplus \mathbf{1}_k(-n)[-2n+1]^{\oplus 2} \hookrightarrow \mathbf{1}_k \oplus \mathbf{1}_k(-n)[-2n+1]^{\oplus 2} \oplus \mathbf{1}_k(-2n)[-4n+2] \simeq \mathbb{F}(\mathbf{1}_k(-n)[-2n+1]^{\oplus 2}) \to M_Y$ is an equivalence. Note that a morphism $\mathbf{1}_k(-2n)[-4n+2] \to \mathbf{1}_k \oplus \mathbf{1}_k(-n)[-2n+1]^{\oplus 2}$ is null homotopic because it corresponds to an element in

$$\begin{aligned}\mathrm{Hom}_{\mathrm{h}(\mathsf{DM}(k))}(\mathbf{1}_k, \mathbf{1}_k(n)[2n-1])^{\oplus 2}) \oplus \mathrm{Hom}_{\mathrm{h}(\mathsf{DM}(k))}(\mathbf{1}_k, \mathbf{1}_k(2n)[4n-2]) \\ \simeq (\mathrm{CH}^n(\mathrm{Spec}\, k, 1)^{\oplus 2} \oplus \mathrm{CH}^{2n}(\mathrm{Spec}\, k, 2)) \otimes_{\mathbb{Z}} \mathbb{Q} = 0\end{aligned}$$

(we use the condition $n \ge 2$). Here $CH^i(-, j)$ is the Bloch's higher Chow group. Hence $V_1 = \mathbf{1}_k(-2n)[-4n+2]$ and $V_1 \to A_1 \simeq \mathbf{1}_k \oplus \mathbf{1}_k(-n)[-2n+1]^{\oplus 2} \oplus \mathbf{1}_k(-2n)[-4n+2]$ may be viewed as the canonical inclusion. We see that

$$A_2 = \mathbb{F}(\mathbf{1}_k(-n)[-2n+1]^{\oplus 2}) \otimes_{\mathbb{F}(\mathbf{1}_k(-2n)[-4n+2])} \mathbf{1}_k.$$

Consider $\mathbb{F}_{\mathrm{Comp}(\mathbb{G}_m)}(\chi_{-2n}[-4n+2]) \to \mathbb{F}_{\mathrm{Comp}(\mathbb{G}_m)}(\chi_{-n}[-2n+1]^{\oplus 2})$ classified by the inclusion $\alpha : \chi_{-2n}[-4n+2] \hookrightarrow \mathbb{F}_{\mathrm{Comp}(\mathbb{G}_m)}(\chi_{-n}[-2n+1]^{\oplus 2})$. Let $\mathbb{F}_{\mathrm{Comp}(\mathbb{G}_m)}(\chi_{-n}[-2n+1]^{\oplus 2})\langle \alpha \rangle$ be the homotopy pushout, see Proposition 5.5. Note that the image of $\mathbb{F}_{\mathrm{Comp}(\mathbb{G}_m)}(\chi_{-n}[-2n+1]^{\oplus 2})\langle \alpha \rangle$ in $\mathrm{CAlg}(\mathsf{DM}^{\otimes}(k))$ (under F) is equivalent to A_2. By the computation using Remark 5.6, we see that

$$\mathbb{F}_{\mathrm{Comp}(\mathbb{G}_m)}(\chi_{-n}[-2n+1]^{\oplus 2})\langle \alpha \rangle \simeq \chi_0 \oplus \chi_{-n}[-2n+1]^{\oplus 2} \oplus \chi_{-3n}[-6n+4]^{\oplus 2} \oplus \chi_{-4n}[-8n+5]$$

in $\mathrm{Rep}(\mathbb{G}_m)$. Hence $A_2 \simeq \mathbf{1}_k \oplus \mathbf{1}_k(-n)[-2n+1]^{\oplus 2} \oplus \mathbf{1}_k(-3n)[-6n+4]^{\oplus 2} \oplus \mathbf{1}_k(-4n)[-8n+5]$. By the argument similar to the case of V_1, we see that $V_2 = \mathbf{1}_k(-3n)[-6n+4]^{\oplus 2} \oplus \mathbf{1}_k(-4n)[-8n+5]$ and $V_2 \to A_2$ may be viewed as the canonical inclusion. We thus find

$$A_3 = A_2 \otimes_{\mathbb{F}(\mathbf{1}_k(-3n)[-6n+4]^{\oplus 2} \oplus \mathbf{1}_k(-4n)[-8n+5])} \mathbf{1}_k.$$

5.1.5 Let $\mathcal{C}^{\otimes}$ be a stable presentable ∞-category endowed with a symmetric monoidal structure whose tensor operation $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves small colimits separately in each variable. Let $\mathbb{F}_{\mathcal{C}} : \mathcal{C} \to \mathrm{CAlg}(\mathcal{C}^{\otimes})$ be the free algebra functor of $\mathcal{C}^{\otimes}$. Let A and B be commutative algebra objects in $\mathrm{CAlg}(\mathcal{C}^{\otimes})$ and $f: A \to B$ be a morphism in $\mathrm{CAlg}(\mathcal{C}^{\otimes})$. Let V be the kernel of f in the stable ∞-category $\mathcal{C}$, i.e., the pullback $A \times_B \{0\}$. Let $\sigma : \mathbb{F}_{\mathcal{C}}(V) \to A$ in $\mathrm{CAlg}(\mathcal{C}^{\otimes})$ be the morphism classified by $V \to A$. Let $\epsilon : \mathbb{F}_{\mathcal{C}}(V) \to \mathbf{1}_{\mathcal{C}} = \mathbb{F}_{\mathcal{C}}(0)$ be the morphism induced by $V \to 0$ where $\mathbf{1}_{\mathcal{C}}$ is the unit algebra in $\mathrm{CAlg}(\mathcal{C}^{\otimes})$. Let us define the commutative algebra object $A(f)$ by the following pushout diagram

$$\begin{array}{ccc} \mathbb{F}_{\mathcal{C}}(V) & \xrightarrow{\sigma} & A \\ {\scriptstyle \epsilon}\downarrow & & \downarrow \\ \mathbf{1}_{\mathcal{C}} & \longrightarrow & A(f) \end{array}$$

in $\mathrm{CAlg}(\mathcal{C}^{\otimes})$. Note that the composite $\mathbb{F}_{\mathcal{C}}(V) \to A \xrightarrow{f} B$ factors through $\mathbb{F}_{\mathcal{C}}(V) \to \mathbf{1}_{\mathcal{C}}$. We have a factorization

$$A \to A(f) \xrightarrow{f'} B$$

of f. Applying this procedure to $f' : A(f) \to B$ we obtain a refined factorization $A \to A(f) \to A(f, f') := A(f)(f') \to B$. Repeating it in the inductive way we have a sequence in $\mathrm{CAlg}(\mathcal{C}^{\otimes})_{/B}$ described as

$$A = A_0 \to A_1 \to A_2 \to \cdots \to A_n \to A_{n+1} \to \cdots$$

where $A_1 = A(f)$, $A_2 = A(f, f') \ldots$. We denote by $f_n : A_n \to B$ the structural morphism. We shall refer to this sequence as the inductive sequence associated to $A \to B$.

Lemma 5.15. *Let* $\varinjlim_n A_n$ *be a colimit of the sequence in* $\mathrm{CAlg}(\mathcal{C}^{\otimes})$. *Then the canonical morphism* $\varinjlim_n A_n \to B$ *is an equivalence in* $\mathrm{CAlg}(\mathcal{C}^{\otimes})$.

Proof. According to [33, 3.2.3.1], the forgetful functor $\mathrm{CAlg}(\mathcal{C}^{\otimes}) \to \mathcal{C}$ preserves filtered colimits. Hence it is enough to prove that a colimit $\varinjlim_n A_n$ in $\mathcal{C}$ (by abuse of notation we continue to use the same symbol) is naturally equivalent to B in $\mathcal{C}$. If V_n denotes the kernel of $f_n : A_n \to B$ in $\mathcal{C}$, then $V_n \to \mathbb{F}_{\mathcal{C}}(V_n) \to A_n \to A_{n+1}$ is null-homotopic. Thus, $A_n \to A_{n+1}$ factors as composition $A_n \to \mathrm{Coker}(V_n \to A_n) \to A_{n+1}$ in $\mathcal{C}$ where $\mathrm{Coker}(-)$ stands for cokernel (cofiber/cone) in $\mathcal{C}$. The sequence $A \to A_1 \to A_2 \to \cdots$ in $\mathcal{C}$ is refined as

$$A \to A_1 \to \mathsf{Coker}(V_1 \to A_1) \to A_2 \to \mathsf{Coker}(V_2 \to A_2) \to A_3 \to \cdots.$$

By cofinality, the colimit of this sequence is naturally equivalent to $\varinjlim_n A_n$. Notice that $\mathrm{Coker}(V_n \to A_n) \simeq B$ in $\mathcal{C}$ for any $n \geq 1$. Hence we deduce that $\varinjlim_n A_n \to B$ is an equivalence in $\mathcal{C}$. □

Remark 5.16. Let $\mathcal{D}^{\otimes}$ be another stable presentable ∞-category endowed with a symmetric monoidal structure whose tensor operation $\mathcal{D} \times \mathcal{D} \to \mathcal{D}$ preserves small colimits separately in each variable. Let $F : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ be a symmetric monoidal functor that preserves small colimits. Our main example of interest is the realization functor $\mathsf{R} : \mathsf{DM}^{\otimes}(k) \to \mathsf{D}^{\otimes}(\mathbb{Q})$. Let $A = A_0 \to A_1 \to \cdots \to B$ be the inductive sequence associated to $f : A \to B$. Note that $\mathcal{C} \to \mathcal{D}$ is an exact functor of stable ∞-categories, and $\mathrm{CAlg}(F) : \mathrm{CAlg}(\mathcal{C}^{\otimes}) \to \mathrm{CAlg}(\mathcal{D}^{\otimes})$ preserves small colimits. Then the sequence $F(A_0) \to F(A_1) \to \cdots \to F(B)$ is canonically equivalent to the inductive sequence associated to $F(A) \to F(B)$ as a diagram in $\mathrm{CAlg}(\mathcal{D}^{\otimes})_{/F(B)}$.

5.2 Let G be a semi-abelian variety over k. There is a (canonical) equivalence

$$M(G) \xrightarrow{\sim} \oplus_{n\geq 0} M_n(G)$$

in $\mathsf{DM}(k)$ such that $M_n(G) = \mathrm{Sym}^n(M_1(G))$. This is a result of Ancona-EnrightWard-Huber [2], which builds upon the works of Shermenev, Deniger-Murre and Künnemann on a decomposition of the motives of an abelian variety, see [29] and references therein. If G is an extension of a g-dimesional abelian variety by a torus of rank r, then $\mathrm{Sym}^n(M_1(G)) \simeq 0$ for $n > 2g + r$. The direct summand $M_1(G)$ is represented, as an object in $\mathrm{Comp}(\mathcal{N}^{tr}(X))$, by the étale sheaf of $\mathbb{Q}$-vector spaces given by $S \mapsto \mathrm{Hom}_{\mathrm{Sm}_k}(S, G) \otimes_{\mathbb{Z}} \mathbb{Q}$ which is promoted to a sheaf with transfers (see e.g. [2, Section 2.1]).

By using their works, we upgrade it to the following:

Proposition 5.17. *Let $M_1(G)^\vee$ be the dual of $M_1(G)$ in $\mathsf{DM}(k)$ ($M_1(G)$ is a dualizable object). Let $\mathbb{F}_{\mathsf{DM}(k)}(M_1(G)^\vee)$ be a free commutative algebra object in $\mathsf{DM}(k)$ generated by $M_1(G)^\vee$. Then there is an equivalence*

$$\mathbb{F}_{\mathsf{DM}(k)}(M_1(G)^\vee) \xrightarrow{\sim} M_G$$

in $\mathrm{CAlg}(\mathsf{DM}^\otimes(k))$.

Remark 5.18. Let $\mathbb{G}$ be a connected compact Lie group. A theorem of Hopf says that there are elements $x_1, \ldots, x_n$ of odd degrees in $H^*(\mathbb{G}, \mathbb{Q})$ such that $H^*(\mathbb{G}, \mathbb{Q})$ is a free commutative graded algebra generated by $x_1, \ldots, x_n$. One can deduce from this theorem that a Sullivan model of $A_{PL}(\mathbb{G})$ is given by a free commutative graded algebra generated by some graded vector space, see [16, Example 3 in page 143]. Proposition 5.17 may be thought of as a generalization of this homotopical statement to $\mathrm{CAlg}(\mathsf{DM}^\otimes(k))$ for semi-abelian varieties.

Lemma 5.19. *Let $\mathcal{C}^\otimes$ be a symmetric monoidal presentable ∞-category whose tensor operation $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves small colimits separately in each variable. Suppose that $\mathcal{C}^\otimes$ is K-linear, namely, it is endowed with a colimit-preserving symmetric monoidal functor $\mathrm{Mod}_K^\otimes \to \mathcal{C}^\otimes$ (K is a field of characteristic zero). Let $\mathrm{h}(\mathcal{C})^\otimes$ be the homotopy category of $\mathcal{C}$ endowed with a symmetric monoidal structure induced by that of $\mathcal{C}^\otimes$. The canonical functor $\pi : \mathcal{C} \to \mathrm{h}(\mathcal{C})$ can be promoted to a symmetric monoidal functor. Let $\pi' : \mathrm{CAlg}(\mathcal{C}^\otimes) \to \mathrm{CAlg}(\mathrm{h}(\mathcal{C})^\otimes)$ be the "projection" induced by the symmetric monoidal functor π. In this Lemma we use the temporary notation $\mathbb{F} := \mathbb{F}_{\mathcal{C}} : \mathcal{C} \to \mathrm{CAlg}(\mathcal{C}^\otimes)$ be a free algebra functor of $\mathcal{C}$. Let $\mathbb{F}^h := \mathbb{F}_{\mathrm{h}(\mathcal{C})} : \mathrm{h}(\mathcal{C}) \to \mathrm{CAlg}(\mathrm{h}(\mathcal{C})^\otimes)$ be a free algebra functor of $\mathrm{h}(\mathcal{C})$. Let $\theta : \mathrm{CAlg}(\mathcal{C}^\otimes) \to \mathcal{C}$ and $\theta^h : \mathrm{CAlg}(\mathrm{h}(\mathcal{C})^\otimes) \to \mathrm{h}(\mathcal{C})$ be forgetful functors. Let C be an object in $\mathcal{C}$. The unit map $C \to \theta(\mathbb{F}(C))$ induces $\pi(C) \to \pi(\theta(\mathbb{F}(C))) = \theta^h(\pi'(\mathbb{F}(C)))$. By the adjunction $(\mathbb{F}^h, \theta^h)$, it gives rise to $\sigma : \mathbb{F}^h(\pi(C)) \to \pi'(\mathbb{F}(C))$. Then the canonical morphism σ is an equivalence.*

Proof. Let $A = \theta^h(\pi'(\mathbb{F}(C)))$. The n-fold multiplication $A^{\otimes n} \to A$ induces $\mathrm{Sym}^n_{\mathrm{h}(\mathcal{C})}(A) \to A$ where $\mathrm{Sym}^n_{\mathrm{h}(\mathcal{C})}(-)$ is the n-fold symmetric product in the K-linear idempotent complete category $\mathrm{h}(\mathcal{C})$. The map $\pi(C) \to A$ induces $\mathrm{Sym}^n_{\mathrm{h}(\mathcal{C})}(\pi(C)) \to \mathrm{Sym}^n_{\mathrm{h}(\mathcal{C})}(A) \to A$. Taking its coproduct we have

$$\tau : \oplus_{n\geq 0} \mathrm{Sym}^n_{\mathrm{h}(\mathcal{C})}(\pi(C)) \to A.$$

Taking account of the canonical equivalence $\oplus_{n\geq 0} \mathrm{Sym}^n_{\mathrm{h}(\mathcal{C})}(\pi(C)) \simeq \mathbb{F}^h(\pi(C))$, it will suffice to show that τ is an isomorphism in $\mathrm{h}(\mathcal{C})$. By [33, 3.1.3.13], there is an equivalence $\oplus_{n\geq 0} \mathrm{Sym}^n_{\mathcal{C}}(C) \to \theta(\mathbb{F}(C))$ where each $\mathrm{Sym}^n_{\mathcal{C}}(C) \to \theta(\mathbb{F}(C))$ is induced by the composition of $C^{\otimes n} \to \mathbb{F}(C)^{\otimes n}$

and the n-fold multiplication $\mathbb{F}(C)^{\otimes n} \to \mathbb{F}(C)$. Here $\mathrm{Sym}^n_{\mathcal{C}}(C)$ is the symmetric product in $\mathcal{C}$. Therefore, it is enough to prove that the natural morphism $\mathrm{Sym}^n_{\mathrm{h}(\mathcal{C})}(\pi(C)) \to \pi(\mathrm{Sym}^n_{\mathcal{C}}(C))$ is an isomorphism. Note that for any D in $\mathcal{C}$, the set $\mathrm{Hom}_{\mathrm{h}(\mathcal{C})}(\mathrm{Sym}^n_{\mathrm{h}(\mathcal{C})}(\pi(C)), \pi(D))$ is the invariant part $\mathrm{Hom}_{\mathrm{h}(\mathcal{C})}(\pi(C)^{\otimes n}, \pi(D))^{\Sigma_n}$ of $\mathrm{Hom}_{\mathrm{h}(\mathcal{C})}(\pi(C)^{\otimes n}, \pi(D))$ with the permutation action of the symmetric group Σ_n. On the other hand, the hom complex $\mathrm{Hom}_{\mathcal{C}}(\mathrm{Sym}^n_{\mathcal{C}}(C), D)$ in Mod_K is a limit $\mathrm{Hom}_{\mathcal{C}}(C^{\otimes n}, D)^{\Sigma_n}$ of $\mathrm{Hom}_{\mathcal{C}}(C^{\otimes n}, D)$ with permutation action of Σ_n. (By definition, the hom complex $\mathrm{Hom}_{\mathcal{C}}(C, D)$ is given by the image of D under the right adjoint $\mathrm{Hom}_{\mathcal{C}}(C, -)$ to the colimit preserving functor $(-) \otimes C : \mathrm{Mod}_K \to \mathcal{C}$.) Since K is a field of characteristic zero (the semi-simplicity of representations of finite groups), we have $H^0(\mathrm{Hom}_{\mathcal{C}}(C^{\otimes n}, D)^{\Sigma_n}) = H^0(\mathrm{Hom}_{\mathcal{C}}(C^{\otimes n}, D))^{\Sigma_n} = \mathrm{Hom}_{\mathrm{h}(\mathcal{C})}(\mathrm{Sym}^n_{\mathrm{h}(\mathcal{C})}(\pi(C)), \pi(D))$. Thus, we see that $\mathrm{Sym}^n_{\mathrm{h}(\mathcal{C})}(\pi(C)) \to \pi(\mathrm{Sym}^n_{\mathcal{C}}(C))$ is an isomorphism. □

Remark 5.20. By the proof, if we define the canonical functor $\pi : \mathsf{DM}(k) \to \mathrm{h}(\mathsf{DM}(k))$, then we have a canonical isomorphism $\mathrm{Sym}^n_{\mathrm{h}(\mathsf{DM}(k))}(\pi(C)) \simeq \pi(\mathrm{Sym}^n_{\mathsf{DM}(k)}(C))$. Namely, π commutes with the formulation of symmetric products. By this canonical isomorphism, we often abuse notation by writing $\mathrm{Sym}^n(C)$ for both $\mathrm{Sym}^n_{\mathrm{h}(\mathsf{DM}(k))}(\pi(C))$ and $\mathrm{Sym}^n_{\mathsf{DM}(k)}(C)$.

Proof of Proposition 5.17. Let $\alpha_G : M(G) \to M_1(G)$ be the morphism described in [2, 2.1.4] (in *loc. cit.*, α_G is a morphism $\mathsf{DM}_{eff}(k)$, but we here regard it as a morphism in $\mathsf{DM}(k)$). We remark also that in [2, 2.1.4] étale motives are empolyed, but $\mathsf{DM}^{\otimes}(k)$ agrees with the étale version since K is a field of characteristic zero, cf. [37], [2, 1.6.1]. Let $\alpha_G^\vee : M_1(G)^\vee \to M(G)^\vee$ be the dual of α_G. Since $M_G = M(G)^\vee$ in $\mathsf{DM}(k)$, $\alpha_G^\vee$ induces a morphism $\mathbb{F}(M_1(G)^\vee) \to M_G$ in $\mathrm{CAlg}(\mathsf{DM}^{\otimes}(k))$. We will prove that it is an equivalence. To see this, it is enough to show that $\pi'(\mathbb{F}(M_1(G)^\vee)) \to \pi'(M_G)$ is an isomorphism where $\pi' : \mathrm{CAlg}(\mathsf{DM}^{\otimes}(k)) \to \mathrm{CAlg}(\mathrm{h}(\mathsf{DM}(k))^{\otimes})$ is the canonical functor (we continue to use the notation in Lemma 5.19). Lemma 5.19 guarantees that $\mathbb{F}^h(\pi(M_1(G)^\vee)) \xrightarrow{\sim} \pi'(\mathbb{F}(M_1(G)^\vee))$. The composite $\mathbb{F}^h(\pi(M_1(G)^\vee)) \to \pi'(M_G)$ is induced by $\pi(M_1(G)^\vee) \to \pi(\theta(M_G)) = \theta^h(\pi'(M_G))$. The proof is reduced to showing that this morphism $\mathbb{F}^h(\pi(M_1(G)^\vee)) \to \pi'(M_G)$ is an isomorphism in $\mathrm{h}(\mathsf{DM}(k))$. The each factor $\phi_n : \mathrm{Sym}^n(\pi(M_1(G)^\vee) \to \pi'(M_G)$ of $\oplus_{n\ge 0}\mathrm{Sym}^n(\pi(M_1(G)^\vee) \xrightarrow{\sim} \mathbb{F}^h(\pi(M_1(G)^\vee)) \to \pi'(M_G)$ is induced by $\pi(M_1(G)^\vee)^{\otimes n} \xrightarrow{(\alpha_G^\vee)^{\otimes n}} \theta^h(\pi'(M_G))^{\otimes n} \to \theta^h(\pi'(M_G))$ where the second morphism is the n-fold multiplication. In the following Lemmata, we will observe that ϕ_n is a dual of the projection $\psi_n : M(G) \to \mathrm{Sym}^n(M_1(G))$ of the equivalence $M(G) \xrightarrow{\sim} \oplus_{0\le n\le 2g+r}\mathrm{Sym}^n(M_1(G))$ proved in [2, Theorem 7.1.1]. It will finish the proof. □

Lemma 5.21. *$\phi_n : \mathrm{Sym}^n(\pi(M_1(G)^\vee)) \to \theta^h(\pi'(M_G))$ is a dual of $\psi_n : M(G) \to \mathrm{Sym}^n(M_1(G))$.*

Proof. We first recall ψ_n. We work with the homotopy category $\mathrm{h}(\mathsf{DM}(k))$. By abuse of notation, we put $M(G) = \pi(M(G))$, $M_1(G) = \pi(M_1(G))$, $\mathrm{Sym}^n(M_1(G)^\vee) = \mathrm{Sym}^n(\pi(M_1(G))^\vee)$, $\mathrm{Sym}^n(M_1(G)) = \mathrm{Sym}^n(\pi(M_1(G)))$, $M_G = \theta^h(\pi'(M_G))$, etc. These idetifications are harmless (cf. Lemma 5.19 and Remark 5.20). The morphism $\psi_n : M(G) \to \mathrm{Sym}^n(M_1(G))$ is the composite $M(G) \to M(G)^{\otimes n} \xrightarrow{\alpha_G^{\otimes n}} M_1(G)^{\otimes n} \to \mathrm{Sym}^n(M_1(G))$ where the first morphism is the n-fold comultiplication and the the third morphism is the canonical projection. By ease of notation, we let $f_\sharp : \mathrm{h}(\mathsf{DM}(G)) \rightleftarrows \mathrm{h}(\mathsf{DM}(k)) : f^*$ be the adjoint pair induced by $f_\sharp : \mathsf{DM}(G) \rightleftarrows \mathsf{DM}(k) : f^*$ where $f : G \to \mathrm{Spec}\, k$ is the structure morphism. The colax monoidal functor $f_\sharp$ induces the coalgebra structure on $M(G) = f_\sharp(\mathbf{1}_X)$ in $\mathrm{h}(\mathsf{DM}(k))$: the comultiplication is given by the composition

$$f_\sharp(\mathbf{1}_G) = f_\sharp(\mathbf{1}_G \otimes \mathbf{1}_G) \to f_\sharp(f^*f_\sharp(\mathbf{1}_G) \otimes f^*f_\sharp(\mathbf{1}_G)) \simeq f_\sharp f^*(f_\sharp(\mathbf{1}_G) \otimes f_\sharp(\mathbf{1}_G)) \to f_\sharp(\mathbf{1}_G) \otimes f_\sharp(\mathbf{1}_G)$$

where the left arrow is induced by the counit of $(f_\sharp, f^*)$, and the right arrow is induced by the unit. The counit $M(G) \to \mathbf{1}_k$ is $f_\sharp f^*(\mathbf{1}_k) \to \mathbf{1}_k$. If one regards $\mathrm{Sym}^n(M_1(G))$ as a direct summand of $M_1(G)^{\otimes n}$, then $M(G) \to M(G)^{\otimes n} \overset{\alpha_G^{\otimes n}}{\to} M_1(G)^{\otimes n}$ factors as $M(G) \overset{\psi_n}{\to} \mathrm{Sym}^n(M_1(G)) \to M(G)^{\otimes n}$. On the other hand, $\phi_n : \mathrm{Sym}^n(M_1(G)^\vee) \to M_G$ is induced by Σ_n-equivariant morphism $(M_1(G)^\vee)^{\otimes n} \overset{(\alpha_G^\vee)^{\otimes n}}{\to} M_G^{\otimes n} \to M_G$ where M_G has the trivial action, and the second arrow is the n-fold multiplication. To prove our assertion of this Lemma, it is enough to show the following general fact:

Lemma 5.22. *Let X be a smooth scheme separated of finite type over k. The multiplication morphism $M_X \otimes M_X \to M_X$ is a dual of the comultiplication morphism $M(X) \to M(X) \otimes M(X)$ given by the diagonal in $\mathrm{h}(\mathsf{DM}(k))$ through the isomorphism $M_X \simeq M(X)^\vee$. (We remark that $M(X)$ is also dualizable in $\mathsf{DM}^{\otimes}(k)$ since we work with coefficients of characteristic zero.)*

Proof. We here write $\mathbf{1} := \mathbf{1}_X$ and the structure morphism $f : X \to \operatorname{Spec} k$. Remember that the multiplication $M_X \otimes M_X \to M_X$ is given by the composition

$$f_*(\mathbf{1}) \otimes f_*(\mathbf{1}) \to f_*f^*(f_*(\mathbf{1}) \otimes f_*(\mathbf{1})) \simeq f_*(f^*f_*(\mathbf{1}) \otimes f^*f_*(\mathbf{1})) \to f_*(\mathbf{1} \otimes \mathbf{1}) \simeq f_*(\mathbf{1})$$

such that the left arrow is induced by the counit of the adjunction (f^*, f_*), and the right arrow is induced by its counit. The canonical isomorphism $\eta : M(X)^\vee \overset{\sim}{\to} M_X$ is defined as follows (see the proof of Proposition 3.4). For $M \in \mathsf{DM}(X)$, consider the unit $M \to f^*f_\sharp(M)$. Taking the dual and f_*, we have $f_*f^*(f_\sharp(M))^\vee \simeq f_*(f^*f_\sharp(M))^\vee \to f_*(M^\vee)$. Composing with the unit $(f_\sharp(M))^\vee \to f_*f^*(f_\sharp(M))^\vee$ we obtain $\eta_M : (f_\sharp(M))^\vee \to f_*(M^\vee)$ which determines an isomorphism $\eta = \eta_{\mathbf{1}_X} : M(X)^\vee \overset{\sim}{\to} M_X$. We will check that the dual of $f_\sharp(\mathbf{1}) \to f_\sharp(\mathbf{1}) \otimes f_\sharp(\mathbf{1})$ is $f_*(\mathbf{1}) \otimes f_*(\mathbf{1}) \to f_*(\mathbf{1})$ through $\eta : f_\sharp(\mathbf{1})^\vee \simeq f_*(\mathbf{1})$. By using the counit of (f^*, f_*) and its counit-unit equations, we see that the dual $f^*f_\sharp(M)^\vee \to M^\vee$ of $M \to f^*f_\sharp(M)$ is $f^*(f_\sharp(M)^\vee) \overset{f^*(\eta_M)}{\to} f^*f_*(M^\vee) \to M^\vee$ where the final arrow is the counit of (f^*, f_*). When $M = \mathbf{1}$, we deduce that the unit $s : \mathbf{1} \to f^*f_\sharp(\mathbf{1})$ is the dual of the counit $t : f^*f_*(\mathbf{1}) \to \mathbf{1}$ through the isomorphism $\eta : f_\sharp(\mathbf{1})^\vee \simeq f_*(\mathbf{1})$. It follows that its tensor product $s \otimes s : \mathbf{1} \otimes \mathbf{1} \to f^*f_\sharp(\mathbf{1}) \otimes f^*f_\sharp(\mathbf{1})$ is the dual of $t \otimes t : f^*f_*(\mathbf{1}) \otimes f^*f_*(\mathbf{1}) \to \mathbf{1} \otimes \mathbf{1}$ through the isomorphism through the isomorphism $\eta : f_\sharp(\mathbf{1})^\vee \simeq f_*(\mathbf{1})$. Thus the triangle in the following diagram commutes.

$$\begin{array}{ccccc}
(f_\sharp(\mathbf{1}) \otimes f_\sharp(\mathbf{1}))^\vee & \xrightarrow{a} & (f_\sharp f^*(f_\sharp(\mathbf{1}) \otimes f_\sharp(\mathbf{1})))^\vee & \xrightarrow{b} & f_\sharp(\mathbf{1})^\vee \\
\simeq \downarrow & & \downarrow \eta_{f^*(f_\sharp(\mathbf{1}) \otimes f_\sharp(\mathbf{1}))} & & \\
f_\sharp(\mathbf{1})^\vee \otimes f_\sharp(\mathbf{1})^\vee & \xrightarrow{c} & f_*f^*(f_\sharp(\mathbf{1})^\vee \otimes f_\sharp(\mathbf{1})^\vee) & \searrow^{f_*(s^\vee \otimes s^\vee)} & \downarrow \eta \\
\eta\otimes\eta \downarrow & & \downarrow f_*f^*(\eta\otimes\eta) & & \\
f_*(\mathbf{1}) \otimes f_*(\mathbf{1}) & \xrightarrow{d} & f_*f^*(f_*(\mathbf{1}) \otimes f_*(\mathbf{1})) & \xrightarrow[f_*(t\otimes t)]{} & f_*(\mathbf{1})
\end{array}$$

Here a is induced by the dual of $f_\sharp f^* \to \mathrm{id}$, and b is induced by the dual of $s \otimes s : \mathbf{1} \otimes \mathbf{1} \to f^*f_\sharp(\mathbf{1}) \otimes f^*f_\sharp(\mathbf{1}) = f^*(f_\sharp(\mathbf{1}) \otimes f_\sharp(\mathbf{1}))$. Note that the composite of the upper horizontal arrows is the dual of comultiplication $M(X) \to M(X) \otimes M(X)$. Both c and d is induced by the unit $\mathrm{id} \to f_*f^*$. The composite of lower horizontal arrows is the multiplication $M_X \otimes M_X \to M_X$. The commutativity of other squares follows from the contravariant functoriality of η_M with respect to M, the functoriality/naturality of $\mathrm{id} \to f_*f^*$, and the counit-unit equations for the adjunction $(f_\sharp, f^*)$. Thus, we have a commutativity of the outer square, which completes the proof. □

Remark 5.23. The unit map $\mathbf{1}_k \to M_X$ is nothing but a dual of the morphism $M(X) \to M(\operatorname{Spec} k) = \mathbf{1}_k$ induced by f.

5.3 Curves

5.3.1 We consider a once-punctured smooth proper curve, that is, $C = \overline{C} - \{p\}$ obtained by removing a k-rational point p from a connected smooth proper curve $\overline{C}$ over the perfect field k. Let $j : C \to \overline{C}$ be the open immersion. The genus of $\overline{C}$ is $g \geq 1$. If $k = \mathbb{C}$, the fundamental group of the underlying topological space $C = \overline{C} - \{p\}$ is the free group generated by $2g$ elements (cf. [49][p. 102]).

Let $M(\overline{C}) \simeq M_0(\overline{C}) \oplus M_1(\overline{C}) \oplus M_2(\overline{C})$ be a (Chow-Künneth) decomposition of $M(\overline{C})$ such that $M_0(\overline{C}) \simeq \mathbf{1}_k$ and $M_2(\overline{C}) \simeq \mathbf{1}_k(1)[2]$ (see the first paragraph of the proof of Lemma 5.28 for the precise formulation). (In this case, $M(C)$ is equivalent to $M_0(\overline{C}) \oplus M_1(\overline{C})$ as an object in $\mathsf{DM}(k)$.) We put $M^i_{\overline{C}} := M_i(\overline{C})^\vee$ so that $M_{\overline{C}} \simeq \oplus_{i=0}^2 M^i_{\overline{C}}$. Let

$$A_0 = \mathbf{1}_k \to A_1 \to A_2 \to \cdots \to A_n \to A_{n+1} \to \cdots$$

be the inductive sequence in $\mathrm{CAlg}(\mathsf{DM}^\otimes(k))_{/M_C}$ associated to the unique morphism $\mathbf{1}_k \to M_C$ in $\mathrm{CAlg}(\mathsf{DM}^\otimes(k))$ (cf. Section 5.1.5). By Lemma 5.15, the colimit $\varinjlim_{n\geq 0} A_n \simeq M_C$.

Theorem 5.24. *We denote by $\mathbb{F}$ the free algebra functor $\mathsf{DM}(k) \to \mathrm{CAlg}(\mathsf{DM}^\otimes(k))$. The first three terms A_1, A_2 and A_3 are computed as follows:*

(1) A_1 is $\mathbb{F}(M^1_{\overline{C}})$, and $f_1 : A_1 \to M_C$ is classified by $M^1_{\overline{C}} \hookrightarrow M^0_{\overline{C}} \oplus M^1_{\overline{C}} \oplus M^2_{\overline{C}} \simeq M_{\overline{C}} \stackrel{j^*}{\to} M_C$.

(2) $f_1 : A_1 \to M_C$ is the composite $M_{\mathrm{Alb}_{\overline{C}}} \stackrel{u^*}{\to} M_{\overline{C}} \stackrel{j^*}{\to} M_C$ up to an equivalence $\mathbb{F}(M^1_{\overline{C}}) \simeq M_{\mathrm{Alb}_{\overline{C}}}$, where $u : \overline{C} \to \mathrm{Alb}_{\overline{C}}$ is the Albanese (Abel-Jacobi) morphism into the Albanese variety, which carries p to the origin.

(3) Let W_1 be $\oplus_{i=2}^{2g} \mathrm{Sym}^i(M^1_{\overline{C}})$. Let $\mathbb{F}(W_1) \to A_1$ be the morphism in $\mathrm{CAlg}(\mathsf{DM}^\otimes(k))$ that is classified by the inclusion $W_1 \to A_1 \simeq \oplus_{i\geq 0} \mathrm{Sym}^i(M^1_{\overline{C}})$ in $\mathsf{DM}(k)$. Then A_2 is the pushout $A_1 \otimes_{\mathbb{F}(W_1)} \mathbf{1}_k$,

(4) Let W_2 be the object in $\mathsf{DM}(k)$ which will be defined just before the proof (Section 5.3.3). Then A_3 has the form of the pushout $A_2 \otimes_{\mathbb{F}(W_2)} \mathbf{1}_k$.

Remark 5.25. The symmetric product $\mathrm{Sym}^N(M^1_{\overline{C}})$ is zero for $N > 2g$ (see the proof of Lemma 5.28). Thus $A_1 = \oplus_{i\geq 0} \mathrm{Sym}^i(M^1_{\overline{C}}) = \oplus_{i=0}^{2g} \mathrm{Sym}^i(M^1_{\overline{C}})$.

Remark 5.26. As mentioned in Introduction, the sequence $\{A_n\}_{n\geq 0}$ can be viewed as a step-by-step description of the "non-abelian structure" of C. To give a feeling for this, let us make the following observation. Suppose that c is a k-rational point on C, and let $M_C \to \mathbf{1}_k$ be the induced augmentation. The sequence $\{A_n\}_{n\geq 0}$ is promoted to a sequence to $\mathrm{CAlg}(\mathsf{DM}^\otimes(k))_{/\mathbf{1}_k}$ in the obvious way. By applying the construction in Section 3.5 to $A_n \to \mathbf{1}_k$, we obtain the sequence of cogroup objects in $\mathrm{CAlg}(\mathsf{DM}^\otimes(k))$, which we denote by $\{\mathbf{1}_k \otimes_{A_n} \mathbf{1}_k\}_{n\geq 0}$ (we abuse notation since $\mathbf{1}_k \otimes_{A_n} \mathbf{1}_k$ is the underlying object). Now consider the "topological aspect" of this sequence. For this purpose, suppose further that $k \subset \mathbb{C}$. Let $R_n \to \mathbb{Q}$ be the image of $A_n \to \mathbf{1}_k$ under the singular realization functor. Let $R \to \mathbb{Q}$ be the image of $M_C \to \mathbf{1}_k$. Then $\{R_n\}_{n\geq 0}$ is the inductive sequence associated to $\mathbb{Q} \to R$ (cf. Remark 5.16), and the image of $\{\mathbf{1}_k \otimes_{A_n} \mathbf{1}_k\}_{n\geq 0}$ under the realization is $\{\mathbb{Q} \otimes_{R_n} \mathbb{Q}\}_{n\geq 0}$ (we abuse notation again). Taking the 0-th cohomology, we have the sequence of the pro-unipotent algebraic groups

$$\cdots \to \operatorname{Spec} H^0(\mathbb{Q} \otimes_{R_n} \mathbb{Q}) \to \cdots \to \operatorname{Spec} H^0(\mathbb{Q} \otimes_{R_1} \mathbb{Q}) \to \operatorname{Spec} H^0(\mathbb{Q} \otimes_{R_0} \mathbb{Q}) \simeq \operatorname{Spec} \mathbb{Q}.$$

Define $G_n := \operatorname{Spec} H^0(\mathbb{Q} \otimes_{R_n} \mathbb{Q})$. In this case, by Theorem 5.24 (i), G_1 is a commutative unipotent group of rank $2g$ (in fact, $\mathbb{F}(M^1_{\overline{C}}[1])$ maps to $\mathbb{Q} \otimes_{R_1} \mathbb{Q} \simeq \mathbb{F}_{\mathbb{Q}}(\mathbb{Q}^{\oplus 2g}) \simeq H^0(\mathbb{Q} \otimes_{R_1} \mathbb{Q})$). Recall that $G := \operatorname{Spec} H^0(\mathbb{Q} \otimes_R \mathbb{Q})$ is the pro-unipotent completion $\pi_1(C^t, c)$ of C^t (cf. Section 7). By a standard argument in rational homotopy theory, each morphism $G_{n+1} \to G_n$ is a surjective morphism with a commutative kernel, and the canonical morphism $\pi_1(C^t, c)_{uni} \simeq G \to \varprojlim_{n \geq 0} G_n$ is an isomorphism of pro-unipotent algebraic groups.

Example 5.27. Let $\overline{C}$ be an elliptic curve and let $C = \overline{C} - \{0\}$ be the open curve obtained by removing the origin 0. Then by Theorem 5.24, one can easily see that $A_1 = \mathbb{F}(M^1_{\overline{C}})$, $A_2 = \mathbb{F}(M^1_{\overline{C}}) \otimes_{\mathbb{F}(\mathbf{1}_k(-1)[-2])} \mathbf{1}_k$, and A_2 is equivalent to $\mathbf{1}_k \oplus M^1_{\overline{C}} \oplus M^1_{\overline{C}}(-1)[-1] \oplus \mathbf{1}_k(-2)[-3]$ as an object in $\mathsf{DM}(k)$. We have $W_2 = M^1_{\overline{C}}(-1)[-1] \oplus \mathbf{1}_k(-2)[-3]$, and the third term A_3 is of the form $A_2 \otimes_{\mathbb{F}(W_2)} \mathbf{1}_k$.

5.3.2

Lemma 5.28. *The multiplication map $M_C \otimes M_C \to M_C$ in the homotopy category $\mathrm{h}(\mathsf{DM}(k))$ is*

$$m : (\mathbf{1}_k \oplus M^1_{\overline{C}})^{\otimes 2} \simeq \mathbf{1}_k \otimes \mathbf{1}_k \oplus \mathbf{1}_k \otimes M^1_{\overline{C}} \oplus M^1_{\overline{C}} \otimes \mathbf{1}_k \oplus (M^1_{\overline{C}})^{\otimes 2} \to \mathbf{1}_k \oplus M^1_{\overline{C}}$$

defined as a coproduct of $m|_{(M^1_{\overline{C}})^{\otimes 2}} = 0$ and "identities" $\mathbf{1}_k \otimes \mathbf{1}_k \to \mathbf{1}_k$, $\mathbf{1}_k \otimes M^1_{\overline{C}} \to M^1_{\overline{C}}$, $M^1_{\overline{C}} \otimes \mathbf{1}_k \to M^1_{\overline{C}}$. Namely, M_C is the trivial square zero extension of $\mathbf{1}_k$ by $M^1_{\overline{C}}$ in $\mathrm{h}(\mathsf{DM}(k))$.

Remark 5.29. The unit $\mathbf{1}_k \to M_C$ may be identified with the morphism $\mathbf{1}_k = M_{\operatorname{Spec} k} \to M_C$ determined by the structure morphism $C \to \operatorname{Spec} k$. By the construction of the decomposition (see below), it is the inclusion $\mathbf{1}_k \hookrightarrow \mathbf{1}_k \oplus M^1_{\overline{C}}$. Thus the non-trivial part of the Lemma is $m|_{(M^1_{\overline{C}})^{\otimes 2}} = 0$.

In the proof of the above Lemma, we discuss decompositions of motives and use the category Chow_k of Chow motives with rational coefficients, cf. [45, Section 1], [40, Section 2.2]. We choose the contravariant Chow motives since we will refer to [45] and [40] in which the authors adopt the contravariant formulation. But $\mathsf{DM}(k)$ is a covariant theory in the sense that there is the canonical covariant functor $\mathrm{Sm}_k \to \mathsf{DM}(k)$ given by $X \mapsto M(X)$ while Chow_k has a contravariant functor $\mathrm{SmPr}_k \to \mathrm{Chow}_k$ given by $X \mapsto \mathsf{ch}(X)$. Here SmPr_k is the category of connected smooth projective varieties over k, and following [40] we denote by $\mathsf{ch}(X)$ the Chow motive of X (that is $h(X)$ in [45]). The relation between $\mathsf{DM}(k)$ and Chow_k is quite well-known, but the difference between the covariant and the contravariant formulations is likely to cause unnecessary confusion. We thus give some remarks. There is a fully faithful $\mathbb{Q}$-linear functor $\mathrm{Chow}_k^{op} \to \mathrm{h}(\mathsf{DM}(k))$ that is symmetric monoidal, [37, 20.1, 20.2], [40, 9.3.6]. It carries $\mathsf{ch}(X)$ to $M(X)$. The Lefschetz motive $\mathbb{L}$ maps to $\mathbf{1}_k(1)[2]$. As the level of hom sets,

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{Chow}_k}(\mathsf{ch}(Y), \mathsf{ch}(X)) = \mathrm{CH}^d(Y \times X) &\overset{\text{transpose}}{\to} \mathrm{CH}^d(X \times Y) \\
&\simeq \mathrm{Hom}_{\mathrm{h}(\mathsf{DM}(k))}(M(X \times Y), \mathbf{1}_k(d)[2d]) \\
&\simeq \mathrm{Hom}_{\mathrm{h}(\mathsf{DM}(k))}(M(X), M(Y)^\vee \otimes \mathbf{1}_k(d)[2d]) \\
&\simeq \mathrm{Hom}_{\mathrm{h}(\mathsf{DM}(k))}(M(X), M(Y))
\end{aligned}$$

where d and e are the dimensions of Y and X, respectively. For $f : X \to Y$, we write $f^* : \mathsf{ch}(Y) \to \mathsf{ch}(X)$ for the class of the transposed graph ${}^t\Gamma_f$ in $\mathrm{CH}^d(Y \times X)$. We also use $f_* : \mathsf{ch}(X) \to \mathsf{ch}(Y) \otimes \mathbb{L}^{\otimes e-d}$ that corresponds to the class of Γ_f in $\mathrm{CH}^{e+(d-e)}(X \times Y)$. The functor

$\mathrm{Chow}_k^{op} \to \mathrm{h}(\mathsf{DM}(k))$ carries $f^*: \mathsf{ch}(Y) \to \mathsf{ch}(X)$ to $M(X) \to M(Y)$ induced by the graph of f, that is the dual of $f^*: M_Y \to M_X$ (the final f^* is defined in Section 3.2).

Proof of Lemma 5.28. For ease of notation, we put $X = \overline{C}$. We first recall the decomposition $\mathsf{ch}^0(X) \oplus \mathsf{ch}^1(X) \oplus \mathsf{ch}^2(X) \simeq \mathsf{ch}(X)$. We define the retract $\mathsf{ch}(\mathrm{Spec}\, k) \xrightarrow{s^*} \mathsf{ch}(X) \xrightarrow{p^*} \mathsf{ch}(\mathrm{Spec}\, k)$ given by $p: \mathrm{Spec}\, k = \{p\} \to X$ and the structure morphism $s: X \to \mathrm{Spec}\, k$. There is also the retract defined by $\mathbb{L} \overset{p_*\otimes\mathbb{L}}{\to} \mathsf{ch}(X) \xrightarrow{s_*} \mathbb{L}$. The components $\mathsf{ch}^0(X) = \mathbf{1}_k$ and $\mathsf{ch}^2(X)$ arise from the first retract and the second retract, respectively. Here we abuse notation by writing $\mathbf{1}_k$ for the unit object in Chow_k because it corresponds to the unit object in $\mathsf{DM}(k)$. The component $\mathsf{ch}^1(X)$ can be described as the Picard motives in the sense of J.P. Murre [40, Section 6.2], [45, Section 4]. Let Alb_X be the Albanese variety of X and let Pic_X be the Picard variety of X. Note that

$$\mathrm{CH}^1(X \times X) \supset \mathrm{Hom}^*((X,p),(\mathrm{Pic}_X,0)) \simeq \mathrm{Hom}_{AV}(\mathrm{Alb}_X, \mathrm{Pic}_X)$$

where Hom^* indicates the set of morphisms that preserve base points, and Hom_{AV} indicates the set of morphisms of abelian varieties. Here we implicitly use the Albanese morphism $(X,p) \to (\mathrm{Alb}_X, 0)$. The set $\mathrm{Hom}^*((X,p),(\mathrm{Pic}_X,0))$ corresponds to the subgroup of $\mathrm{CH}^1(X\times X)$, that consists of those classes of divisors $D \in \mathrm{CH}^1(X \times X)$ such that $(\mathrm{id}_X \times p)^*(D) = 0$ and $(p \times \mathrm{id}_X)^*(D) = 0$ in $\mathrm{CH}^1(X)$. We will call such divisors p-normalized divisors and denote by $\mathrm{CH}^1_{(p)}(X \times X)$ the subgroup of p-normalized divisors. Consider the isomorphism $\theta: \mathrm{Alb}_X \xrightarrow{\sim} \mathrm{Pic}_X$ defined by the theta divisor. By [40, Lemma 6.2.6] the element π_1 in $\mathrm{Hom}_{\mathrm{Chow}_k}(\mathsf{ch}(X), \mathsf{ch}(X)) = \mathrm{CH}^1(X \times X) \otimes_{\mathbb{Z}} \mathbb{Q}$ corresponding to θ is an idempotent morphism of $\mathsf{ch}(X)$. We define $\mathsf{ch}^1(X)$ to be the object corresponding to π_1, namely, $\mathsf{ch}^1(X) \to \mathsf{ch}(X)$ is $\mathrm{Ker}(\mathrm{id} - \pi_1) \to \mathsf{ch}(X)$. Let $M_0(X) \oplus M_1(X) \oplus M_2(X) \simeq M(X)$ be the decomposition that arises from the decomposition of $\mathsf{ch}(X) \simeq \mathsf{ch}^1(X) \oplus \mathsf{ch}^1(X) \oplus \mathsf{ch}^2(X)$. Put $M_X^i = M_i(X)^\vee$ and let $M_X^0 \oplus M_X^1 \oplus M_X^2 \simeq M_X$ be the decomposition obtained by taking the dual. We remark that $M_X^0 \simeq \mathbf{1}_k$ and $M_X^2 \simeq \mathbf{1}_k(-1)[-2]$.

Next we construct a Picard motive $\mathsf{ch}^1(\mathrm{Alb}_X)$ by using the Albanese (Abel-Jacobi) map $u: X \to \mathrm{Alb}_X$ which carries p to the origin. Consider the isomorphisms $\mathrm{Pic}_{\mathrm{Alb}_X} \xrightarrow{\sim} \mathrm{Pic}_X \xleftarrow{\theta} \mathrm{Alb}_X \xrightarrow{\sim} \mathrm{Alb}_{\mathrm{Alb}_X}$. The third morphism induced by the functoriality is an isomorphism because of the universal property of Albanese vatieties, and the first morphism is its dual. Let $\sigma: \mathrm{Alb}_{\mathrm{Alb}_X} \to \mathrm{Pic}_{\mathrm{Alb}_X}$ be the inverse of the composite. If we denote by $\mathrm{CH}^1_{(0)}(\mathrm{Alb}_X \times \mathrm{Alb}_X)$ the subgroup of 0-normalized divisors (0 is the origin), we have the canonical isomorphisms $\mathrm{Hom}_{AV}(\mathrm{Alb}_{\mathrm{Alb}_X}, \mathrm{Pic}_{\mathrm{Alb}_X}) \simeq \mathrm{Hom}_{AV}(\mathrm{Alb}_X, \mathrm{Pic}_{\mathrm{Alb}_X}) \simeq \mathrm{CH}^1_{(0)}(\mathrm{Alb}_X \times \mathrm{Alb}_X)$. Let Z be the divisor that corresponds to σ and let $\phi: \mathsf{ch}(\mathrm{Alb}_X) \otimes \mathbb{L}^{\otimes 1-g} \to \mathsf{ch}(\mathrm{Alb}_X)$ be the morphism defined by $Z \in \mathrm{CH}^1(\mathrm{Alb}_X \times \mathrm{Alb}_X)$. Let $\omega_1: \mathsf{ch}(\mathrm{Alb}_X) \to \mathsf{ch}(\mathrm{Alb}_X)$ be the composite

$$\mathsf{ch}(\mathrm{Alb}_X) \xrightarrow{u^*} \mathsf{ch}(X) \xrightarrow{u_*} \mathsf{ch}(\mathrm{Alb}_X) \otimes \mathbb{L}^{\otimes 1-g} \xrightarrow{\phi} \mathsf{ch}(\mathrm{Alb}_X).$$

We can apply the proof of [40, Lemma 6.2.6] and see that ω_1 is an idempotent map. We define $\mathsf{ch}^1(\mathrm{Alb}_X)$ to be $\mathrm{Ker}(\mathrm{id} - \omega_1)$.

We now claim that the composite

$$\mathsf{ch}^1(\mathrm{Alb}_X) \to \mathsf{ch}(\mathrm{Alb}_X) \xrightarrow{u^*} \mathsf{ch}(X) \to \mathsf{ch}^1(X)$$

is an isomorphism where the first arrow is the canonical monomorphism and the final arrow is the "projection". As observed in [40, proof of Lemma 6.2.6], the equality $(E): \phi\circ u_*\circ u^*\circ\phi\circ u_* = \phi\circ u_*$ holds (indeed, $\omega_1\circ\omega_1 = \omega_1$ is a direct consequence of (E)). Thus, $u^*\circ\phi\circ u_*\circ u^*\circ\phi\circ u_* = u^*\circ\phi\circ u_*$.

Namely, $u^*\circ\phi\circ u_* : \mathsf{ch}(X)\to\mathsf{ch}(X)$ is an idempotent morphism. The morphism π_1 coincides with $u^*\circ\phi\circ u_*$. Actually, again by the observation in [40, proof of Lemma 6.2.6], $u^*\circ\phi\circ u_*$ corresponds to the composite $\mathrm{Alb}_X \stackrel{\sim}{\to} \mathrm{Alb}_{\mathrm{Alb}_X} \stackrel{\sigma}{\to} \mathrm{Pic}_{\mathrm{Alb}_X} \stackrel{\sim}{\to} \mathrm{Pic}_X$, that is θ, through $\mathrm{CH}^1(X\times X)\supset \mathrm{Hom}^*((X,p),(\mathrm{Pic}_X,0)) \simeq \mathrm{Hom}_{AV}(\mathrm{Alb}_X,\mathrm{Pic}_X)$. Let $F := (u^*\circ\phi\circ u_*)\circ u^*\circ(\phi\circ u_*\circ u^*)$ and $G := (\phi\circ u_*\circ u^*)\circ\phi\circ u_*\circ(u^*\circ\phi\circ u_*)$. To prove our claim, it will suffice to show $F\circ G = u^*\circ\phi\circ u_*$ and $G\circ F = \phi\circ u_*\circ u^*$. These equalities follow from (E). We also see that $\phi\circ u_*$ induces $\mathsf{ch}^1(X)\to\mathsf{ch}(X)\stackrel{\phi\circ u_*}{\to}\mathsf{ch}(\mathrm{Alb}_X)\to\mathsf{ch}^1(\mathrm{Alb}_X)$ is an isomorphism.

Let $\mathbb{F}_{\mathrm{Chow}_k}(\mathsf{ch}^1(\mathrm{Alb}_X)) = \oplus_{n\ge 0}\mathrm{Sym}^n(\mathsf{ch}^1(\mathrm{Alb}_X))$ be the free commutative algebra object in Chow_k and let $h : \mathbb{F}_{\mathrm{Chow}_k}(\mathsf{ch}^1(\mathrm{Alb}_X)) \to \mathsf{ch}(\mathrm{Alb}_X)$ be the morphism of commutative algebra objects that is classified by $\mathsf{ch}^1(\mathrm{Alb}_X)\to\mathsf{ch}(\mathrm{Alb}_X)$. Here $\mathsf{ch}(\mathrm{Alb}_X)$ admits the commutative algebra structure defined by $\mathsf{ch}(\mathrm{Spec}\,k)\to\mathsf{ch}(\mathrm{Alb}_X)$ induced by the structure morphism and $\mathsf{ch}(X)\otimes\mathsf{ch}(X)\to\mathsf{ch}(X)$ induced by the diagonal. We will show that h is an isomorphism. Let $R_l : \mathrm{Chow}_k \to \mathrm{GrVect}$ be the (symmetric monoidal) l-adic realization functor to the category of $\mathbb{Z}$-graded $\mathbb{Q}_l$-vector space (the symmetric monoidal structure on GrVect adopts the Koszul rule). For a projective smooth variety U, it carries $\mathsf{ch}(U)$ to the $\mathbb{Z}$-graded $\mathbb{Q}_l$-vector space $\mathrm{H}^*_{\text{ét}}(U\times_k\overline{k},\mathbb{Q}_l)$ of étale cohomology ($\overline{k}$ is a separable closure). Then $\mathrm{H}^*_{\text{ét}}(\mathrm{Alb}_X\times_k\overline{k},\mathbb{Q}_l)$ is the free commutative graded algebra generated by $\mathrm{H}^1_{\text{ét}}(\mathrm{Alb}_X\times_k\overline{k},\mathbb{Q}_l)$ placed in degree one. By [40, Theorem 6.2.1], $R_l(\mathsf{ch}^1(\mathrm{Alb}_X))$ is $\mathrm{H}^1_{\text{ét}}(\mathrm{Alb}_X\times_k\overline{k},\mathbb{Q}_l)$ placed in degree one, and $R_l(\mathsf{ch}^1(\mathrm{Alb}_X)\to\mathsf{ch}(\mathrm{Alb}_X))$ is $\mathrm{H}^1_{\text{ét}}(\mathrm{Alb}_X\times_k\overline{k},\mathbb{Q}_l)\hookrightarrow\mathrm{H}^*_{\text{ét}}(\mathrm{Alb}_X\times_k\overline{k},\mathbb{Q}_l)$. We then conclude that $R_l(h)$ is an isomorphism. Since $\mathrm{Sym}^N(\mathsf{ch}^1(\mathrm{Alb}_X)) = \mathrm{Sym}^N(\mathsf{ch}^1(X)) = 0$ for $N>2g$ (see e.g. [29]), $\mathbb{F}_{\mathrm{Chow}_k}(\mathsf{ch}^1(\mathrm{Alb}_X)) = \oplus_{n=0}^{2g}\mathrm{Sym}^n(\mathsf{ch}^1(\mathrm{Alb}_X))$. Both $\mathsf{ch}(\mathrm{Alb}_X)$ and $\mathbb{F}_{\mathrm{Chow}_k}(\mathsf{ch}^1(\mathrm{Alb}_X))$ are Kimura finite (see e.g. [40] for this notion). Thanks to André-Kahn [4, Proposition 1.4.4.(b), Theorem 9.2.2] (explained also in [2, Theorem 1.3.1]), we deduce from the isomorphism $R_l(h)$ that h is an isomorphism. (We remark that h is not necessarily compatible with the equivalence in Proposition 5.17.)

Next consider the composition

$$\psi : \mathbb{F}_{\mathrm{Chow}_k}(\mathsf{ch}^1(\mathrm{Alb}_1)) = \oplus_{i=0}^{2g}\mathrm{Sym}^i(\mathsf{ch}^1(\mathrm{Alb}_X)) \stackrel{h}{\simeq} \mathsf{ch}(\mathrm{Alb}_X) \stackrel{u^*}{\to} \mathsf{ch}(X) \stackrel{\pi_1}{\to} \mathsf{ch}^1(X).$$

We will show that for $i\neq 1$, $\mathrm{Sym}^i(\mathsf{ch}^1(\mathrm{Alb}_X))\hookrightarrow\mathsf{ch}(\mathrm{Alb}_X)\stackrel{\psi}{\to}\mathsf{ch}^1(X)$ is zero. Let $\omega_i : \mathsf{ch}(\mathrm{Alb}_X)\to\mathrm{Sym}^i(\mathsf{ch}^1(\mathrm{Alb}_X))\to\mathsf{ch}(\mathrm{Alb}_X)$ denote the idempotent map arising from the summand $\mathrm{Sym}^i(\mathsf{ch}^1(\mathrm{Alb}_X))$. Note that $\omega_1 : \mathsf{ch}(\mathrm{Alb}_X)\to\mathsf{ch}(\mathrm{Alb}_X)$ equals to

$$\mathsf{ch}(\mathrm{Alb}_X) \stackrel{u^*}{\to} \mathsf{ch}(X) \stackrel{\pi_1}{\to} \mathsf{ch}^1(X) \hookrightarrow \mathsf{ch}(X) \stackrel{\phi\circ u_*}{\to} \mathsf{ch}(\mathrm{Alb}_X).$$

Indeed, the equality (E) implies that $\phi\circ u_*\circ\pi_1\circ u^* = \phi\circ u_*\circ u^* = \omega_1$. Suppose that $\mathrm{Sym}^i(\mathsf{ch}(\mathrm{Alb}_X))\to\mathsf{ch}^1(X)$ induced by ψ is not zero. It follows that $w_1\circ w_i$ is not zero because $\phi\circ u_*$ induces the isomorphism $\mathsf{ch}^1(X)\simeq\mathsf{ch}^1(\mathrm{Alb}_X)\subset\mathsf{ch}(\mathrm{Alb}_X)$. For $i\neq 1$, this contradicts the orthogonality $w_1\circ w_i = 0$. Hence $\mathrm{Sym}^i(\mathsf{ch}(\mathrm{Alb}_X))\to\mathsf{ch}^1(X)$ is zero for $i\neq 1$. Remember that $\mathsf{ch}(X)$ has a commutative algebra structure (in Chow_k) that is defined by the structure morphism and the diagonal in the same way as $\mathsf{ch}(\mathrm{Alb}_X)$. In addition, u^* is a homomorphism of commutative algebras. The homomorphism u^* induces an isomorphism $\mathsf{ch}^1(\mathrm{Alb}_X)\simeq\mathsf{ch}^1(X)$. Taking account of the compatibility of multiplications, we see that the multiplication $\mathsf{ch}(X)\otimes\mathsf{ch}(X)\to\mathsf{ch}(X)\simeq\mathsf{ch}^1(X)\oplus\mathsf{ch}^1(X)\oplus\mathsf{ch}^2(X)$ sends $\mathsf{ch}^1(X)\otimes\mathsf{ch}^1(X)$ to the direct summand $\mathsf{ch}^2(X)\subset\mathsf{ch}(X)$. Namely, the composition $\mathsf{ch}^1(X)\otimes\mathsf{ch}^1(X)\hookrightarrow\mathsf{ch}(X)\otimes\mathsf{ch}(X)\to\mathsf{ch}(X)\to\mathsf{ch}^1(X)$ is zero (by the compatibility of the unit maps the projection to $\mathsf{ch}^0(X)$ is also zero). Now move to $\mathrm{h}(\mathsf{DM}(k))$. The commutative algebra $\mathsf{ch}(X)$ corresponds to the cocommutative coalgebra $M(X)$ whose coalgebra structure is determined by the structure morphism and the diagonal. Take the dual of $M(X)$, that is, M_X in $\mathrm{h}(\mathsf{DM}(k))$. According to Lemma 5.22, the algebra structure of M_X (in $\mathrm{h}(\mathsf{DM}(k))$) given by the coalgebra structure of $M(X)$ coincides with that of M_X induced by the cohomological motivic algebra. We have proved that the multiplication $M_X \otimes M_X \to M_X$ sends the component $M_X^1 \otimes M_X^1$ to the direct summand M_X^2. If $j : C \to X$ denote the open immersion, then we have the dual of the Gysin distinguished triangle [37, 14.5]

$$\mathbf{1}_k(-1)[-2] \xrightarrow{\eta} M_X \xrightarrow{j^*} M_C \to$$

in $\mathrm{h}(\mathsf{DM}(k))$. By the exact sequence $\mathrm{CH}^0(\operatorname{Spec} k) \xrightarrow{p_*} \mathrm{CH}^1(X) \xrightarrow{j^*} \mathrm{CH}^1(C) \to 0$, we see that the composite $\mathbf{1}_k(-1)[-2] \xrightarrow{e} M_X \xrightarrow{j^*} M_C$ is zero where e is the dual of the morphism $M(X) \to \mathbf{1}_k(1)[2]$ corresponding to the class of p in $\mathrm{CH}^1(X)$. Since e is non-trivial and

$$\mathrm{End}_{\mathrm{h}(\mathsf{DM}(k))}(\mathbf{1}_k(-1)[-2]) = \mathbb{Q}$$

thus we may suppose that $e = \eta$. Consequently, η is the canonical inclusion $\mathbf{1}_k(-1)[-2] \simeq M_X^2 \hookrightarrow M_X^0 \oplus M_X^1 \oplus M_X^2$. It follows that j^* is identified with the projection $M_X \simeq M_X^0 \oplus M_X^1 \oplus M_X^2 \to M_X^0 \oplus M_X^1 \simeq M_C$ (with respect this decomposition). Therefore, the multiplication $M_C \otimes M_C \to M_C$ sends the component $M_X^1 \otimes M_X^1 \subset M_C \otimes M_C$ to zero. □

Remark 5.30. One can ask whether or not M_C is a trivial square zero extension of $\mathbf{1}_k$ by some motive M in $\mathsf{DM}(k)$ (not only at the level of $\mathrm{h}(\mathsf{DM}(k))$). It would be an interesting problem. We refer to [33, 7.3.4] for the notion of trivial square zero extensions in ∞-categorical setting. If M_C is the trivial square zero extension at the level of $\mathsf{DM}(k)$, it should be regarded as *formality* of M_C. Suppose that we are given a connected affine smooth curve C over $\mathbb{C}$. Write C^t for the underlying topological space of C. Then $A_{PL}(C^t)$ is equivalent to the square zero extension $\mathbb{Q} \oplus H^1(C^t, \mathbb{Q})[-1]$ of $\mathbb{Q} = H^0(C^t, \mathbb{Q})$ by $H^1(C^t, \mathbb{Q})[-1]$ in $\mathrm{CAlg}_{\mathbb{Q}}$. (Namely, $A_{PL}(C^t)$ is formal.) The problem about formality of M_C makes sense for arbitrary (geometrically connected) affine smooth curves.

5.3.3 Before the proof of Theorem 5.24, we will define W_2. Let K be the standard representation of GL_{2g}, that is, the $2g$-dimensional vector space V endowed with the canonical action of $\mathrm{Aut}(V) = \mathrm{GL}_{2g}$. We usually consider K to be the complex concentrated in degree zero, that belongs to either $\mathrm{Comp}(\mathrm{GL}_{2g})$ or $\mathrm{Rep}(\mathrm{GL}_{2g})$. Let $\mathbb{F}_{\mathrm{Comp}(\mathrm{GL}_{2g})}(K[-1])$ is the free commutative algebra object in $\mathrm{Comp}(\mathrm{GL}_{2g})$, that is isomorphic to $\oplus_{i=0}^{2g} \mathrm{Sym}^i(K[-1])$ as an object of $\mathrm{Comp}(\mathrm{GL}_{2g})$. We put $U_1 = \oplus_{i=2}^{2g} \mathrm{Sym}^i(K[-1])$ and consider the inclusion $\alpha : U_1 \hookrightarrow \oplus_{i=0}^{2g} \mathrm{Sym}^i(K[-1])$. We let $\phi_\alpha : \mathbb{F}_{\mathrm{Comp}(\mathrm{GL}_{2g})}(U_1) \to \mathbb{F}_{\mathrm{Comp}(\mathrm{GL}_{2g})}(K[-1])$ be the morphism classified by α and let $\mathbb{F}_{\mathrm{Comp}(\mathrm{GL}_{2g})}(K[-1])\langle\alpha\rangle$ be the homotopy pushout (cf. Proposition 5.5). Consider $\mathbb{F}_{\mathrm{Comp}(\mathrm{GL}_{2g})}(K[-1])\langle\alpha\rangle$ as an object in $\mathrm{Comp}(\mathrm{GL}_{2g})$. Then by the explicit presentation in Remark 5.6, we find that its 0-th cohomogy is the unit, and the first cohomology is K. Thus, $\mathbb{F}_{\mathrm{Comp}(\mathrm{GL}_{2g})}(K[-1])\langle\alpha\rangle$ is equivalent to $\mathbf{1} \oplus K[-1] \oplus U_2$ in $\mathrm{Rep}(\mathrm{GL}_{2g})$ where $\mathbf{1}$ is a unit object in $\mathrm{Rep}(\mathrm{GL}_{2g})$, and U_2 is concentrated in the degrees larger than one. (We remark that in practice one can compute U_2 explicitly by means of the representation theory of GL_{2g}.) Note that the wedge product $\wedge^N(M_{\overline{C}}^1[1]) = (\mathrm{Sym}^N(M_{\overline{C}}^1))[N]$ is zero exactly when $N > 2g$ because $M_{\overline{C}}^1$ is equivalent to the dual of the direct summand $M_1(\mathrm{Alb}_{\overline{C}})$ arising from $\mathsf{ch}^1(\mathrm{Alb}_{\overline{C}})$ (see the proof of Lemma 5.28), and $\mathrm{Sym}^i(M_1(\mathrm{Alb}_{\overline{C}})^\vee) = 0$ for $N > 2g$, see e.g. [29] for this vanishing. Thanks to Proposition 5.2, there is a symmetric monoidal colimit-preserving functor $F : \mathrm{Rep}^\otimes(\mathrm{GL}_{2g}) \to \mathsf{DM}^\otimes(k)$ which carries K to $M_{\overline{C}}^1[1]$. We define W_2 to be $F(U_2)$.

Proof of Theorem 5.24. We first prove (1). For simplicity, we put $X = \overline{C}$. Taking account of the construction of the decomposition $\mathsf{ch}(C) \simeq \mathsf{ch}^0(X) \oplus \mathsf{ch}^1(X)$ in the proof of Lemma 5.28, $\mathbf{1}_k = M_{\mathrm{Spec}\, k} \to M_C$ induced by the structure morphism $C \to \mathrm{Spec}\, k$ is identified with the inclusion $\mathbf{1}_k = M_X^0 \hookrightarrow M_X^0 \oplus M_X^1$. Note that the unit algebra $\mathbf{1}_k$ is an initial object. Thus, the kernel of $A_0 = \mathbf{1}_k \to M_C$ in $\mathsf{DM}(k)$ is $M_X^1[-1]$, that is, $V_0 = M_X^1[-1]$. Therefore $A_1 = \mathbf{1}_k \otimes_{\mathbb{F}(M_X^1[-1])} \mathbf{1}_k \simeq \mathbb{F}(0 \sqcup_{M_X^1[-1]} 0) \simeq \mathbb{F}(M_X^1)$ ($\sqcup$ indicates the pushout). The composite $M_X \to \mathbb{F}(M_X^1) \to M_C$ is equivalent to the inclusion $M_X^1 \simeq 0 \sqcup_{M_X^1[-1]} 0 \to \mathbf{1}_k \sqcup_{M_X^1[-1]} 0 \simeq \mathbf{1}_k \oplus M_X^1$ where the second arrow is induced by $0 \to \mathbf{1}_k$. Next we prove (2). Let $M(\mathrm{Alb}_X) \to M_1(\mathrm{Alb}_X)$ be the morphism arising from $\mathsf{ch}^1(\mathrm{Alb}_X) \to \mathsf{ch}(\mathrm{Alb}_X)$ (see the second paragraph of the proof of Lemma 5.28). If one takes its dual $M_{\mathrm{Alb}_X}^1 = M_1(\mathrm{Alb}_X)^\vee \to M_{\mathrm{Alb}_X}$, then by Lemma 5.19, the induced map $\mathbb{F}(M_{\mathrm{Alb}_X}^1) \to M_{\mathrm{Alb}_X}$ is an equivalence in $\mathrm{CAlg}(\mathsf{DM}^\otimes(k))$. By the isomorphism $\mathsf{ch}^1(\mathrm{Alb}_X) \simeq \mathsf{ch}^1(X)$ in the third paragraph of the proof of Lemma 5.28, the composite $\mathbb{F}(M_{\mathrm{Alb}_X}^1) \to M_{\mathrm{Alb}_X} \overset{u^*}{\to} M_X \overset{j^*}{\to} M_C$ induces an equivalence $M_{\mathrm{Alb}_X}^1 \to \mathbb{F}(M_{\mathrm{Alb}_X}^1) \to M_C \to M_X^1$. Also, $M_{\mathrm{Alb}_X}^1 \to M_C \to M_X^0$ is null homotopic. Consider $\mathbb{F}(M_X^1) \simeq \mathbb{F}(M_{\mathrm{Alb}_X}^1)$ induced by $\mathsf{ch}^1(X) \simeq \mathsf{ch}^1(\mathrm{Alb}_X)$. Then $\mathbb{F}(M_X^1) \simeq \mathbb{F}(M_{\mathrm{Alb}_X}^1) \simeq M_{\mathrm{Alb}_X} \overset{j^*u^*}{\to} M_C$ is equivalent to $A_1 \to M_C$.

Next we prove (3). Let V_1 is the kernel of $A_1 = \mathbb{F}(M_X^1) \to M_C$. Then $M_X^1 \to \mathbb{F}(M_X^1) \to M_C \simeq \mathbf{1}_k \oplus M_X^1$ may be viewed as the inclusion. In addition, Lemma 5.28 shows that $M_C \otimes M_C \to M_C$ kills $M_X^1 \otimes M_X^1$. Thus, taking account of the commutative algebra structure of $\mathbb{F}(M_X^1)$ in $\mathrm{h}(\mathsf{DM}(k))$ we deduce that $\mathbb{F}(M_X^1) = \oplus_{i=0}^{2g} \mathrm{Sym}^i(M_X^1) \to M_C \simeq \mathbf{1}_k \oplus M_X^1$ can be identified with the projection. Hence $V_1 \to \mathbb{F}(M_X^1)$ is $\oplus_{i=2}^{2g} \mathrm{Sym}^i(M_X^1) \hookrightarrow \oplus_{i=0}^{2g} \mathrm{Sym}^i(M_X^1)$. Let $\mathbb{F}(V_1) \to \mathbb{F}(M_X^1)$ is the morphism classified by $V_1 \to \mathbb{F}(M_X^1)$. Thus $A_2 = \mathbb{F}(M_X^1) \otimes_{\mathbb{F}(V_1)} \mathbf{1}_k$.

Next we prove the assertion (4). Note that we already defined an "explicit" model of A_2 before this proof. Namely, A_2 is equivalent to the image of $\mathbb{F}_{\mathrm{Comp}(\mathrm{GL}_{2g})}(K[-1])\langle\alpha\rangle$ under $\mathrm{CAlg}(F)$: $\mathrm{CAlg}(\mathrm{Rep}^\otimes(\mathrm{GL}_{2g})) \to \mathrm{CAlg}(\mathsf{DM}^\otimes(k))$. Thus $A_2 \simeq \mathbf{1}_k \oplus M_X^1 \oplus W_2$. Moreover, using the sequence $A_1 \to A_2 \to M_C$ we find that the composite $r : \mathbf{1}_k \oplus M_X^1 \hookrightarrow \mathbf{1}_k \oplus M_X^1 \oplus W_2 \simeq A_2 \to M_C \simeq \mathbf{1}_k \oplus M_X^1$ is an equivalence. Put $h : W_2 \hookrightarrow \mathbf{1}_k \oplus M_X^1 \oplus W_2 \simeq A_2 \to M_C \simeq \mathbf{1}_k \oplus M_X^1$. Then $H = (-r^{-1} \circ h) \oplus \mathrm{id}_{W_2} : W_2 \to (\mathbf{1}_k \oplus M_X^1) \oplus W_2 \simeq A_2$ is the kernel of $A_2 \to M_C$ (we expect that h is zero). Let $\mathbb{F}(W_2) \to A_2$ be the morphism classified by H. Then $A_3 = A_2 \otimes_{\mathbb{F}(W_2)} \mathbf{1}_k$. □

6. Cotangent motives and homotopy groups

We introduce a cotangent motive of a pointed (smooth) scheme (X, x). Under a suitable condition, the dual of rational homotopy groups will appear as the realization of the cotangent motive. The notion of cotagent motives is inspired by Sullivan's description of rational homotopy groups in terms of the space of indecomposable elements of a minimal Sullivan model. We may think of cotangent motive as motives of (dual of) rational homotopy groups. In this section, the coefficient ring of $\mathsf{DM}(k)$ is $\mathbb{Q}$.

6.1 Let $(X, x : \mathrm{Spec}\, k \to X)$ be a pointed smooth scheme over k. It gives rise to an augmented object $x^* : M_X \to \mathbf{1}_k = M_{\mathrm{Spec}\, k}$. We will define an object of $\mathsf{DM}(k)$ by means of cotangent complexes for $\mathrm{CAlg}(\mathsf{DM}^\otimes(k))$. For this purpose, we use the theory of cotangent complexes for presentable ∞-categories, developed in [33, Section 7.3]. This theory is a vast generalization of cotangent complexes (topological André-Quillen homology) for E_∞-algebras. Let us briefly recall some definitions about cotangent complexes for the reader's convenience. Let $\mathcal{C}$ be a presentable ∞-category and let A be an object in $\mathcal{C}$. Let $\mathrm{Sp}(\mathcal{C}_{/A})$ be the stabilization (stable envelope) of

$\mathcal{C}_{/A}$ (cf. [33, 1.4]). Let $(\mathcal{C}_{/A})_*$ denote the ∞-category of pointed objects of $\mathcal{C}_{/A}$: one may take $(\mathcal{C}_{/A})_* = (\mathcal{C}_{/A})_{A/}$. Then $\mathrm{Sp}(\mathcal{C}_{/A})$ is defined to be the limit of the sequence of ∞-categories

$$\cdots \stackrel{\Omega_*}{\to} (\mathcal{C}_{/A})_* \stackrel{\Omega_*}{\to} (\mathcal{C}_{/A})_* \stackrel{\Omega_*}{\to} (\mathcal{C}_{/A})_*,$$

where Ω_* is informally given by $S \mapsto * \times_S *$ ($*$ is a final object). The stable ∞-category $\mathrm{Sp}(\mathcal{C}_{/A})$ is also presentable. Another presentation of $\mathrm{Sp}(\mathcal{C}_{/A})$ is given by the ∞-category of spectrum objects of $\mathcal{C}$ relative to $(\mathcal{C}_{/A})_*$, see [33, 1.4.2]. There is a canonical functor $\Omega^\infty : \mathrm{Sp}(\mathcal{C}_{/A}) \to (\mathcal{C}_{/A})_* \to \mathcal{C}_{/A}$ where the first arrow is the projection to $(\mathcal{C}_{/A})_*$ placed in the right end in the above sequence. The second arrow is the forgetful functor. Let $\Sigma^\infty_+ : \mathcal{C}_{/A} \to \mathrm{Sp}(\mathcal{C}_{/A})$ be a left adjoint to Ω^∞, whose existence is ensured by adjoint functor theorem since Ω^∞ preserves small limits and is accessible. An absolute cotangent complex L_A of A is defined to be $\Sigma^\infty_+(A \stackrel{\mathrm{id}}{\to} A)$. If A is an initial object, then L_A is a zero object. We now take $\mathcal{C}$ to be $\mathrm{CAlg}(\mathsf{DM}^\otimes(k))$. By [33, 7.3.4.13], there is a canonical equivalence $\mathrm{Sp}(\mathrm{CAlg}(\mathsf{DM}^\otimes(k))_{/A}) \simeq \mathrm{Mod}_A(\mathsf{DM}(k))$ of ∞-categories. Here $\mathrm{Mod}_A(\mathsf{DM}(k))$ denotes the ∞-category of A-module objects in $\mathsf{DM}(k)$. We refer to [33, 3.3.3, 4.5] for the notion of module objects over a commutative algebra object. We have the adjunction

$$\Sigma^\infty_+ : \mathrm{CAlg}(\mathsf{DM}^\otimes(k))_{/A} \rightleftarrows \mathrm{Sp}(\mathrm{CAlg}(\mathsf{DM}^\otimes(k))_{/A}) \simeq \mathrm{Mod}_A(\mathsf{DM}(k)) : \Omega^\infty.$$

We regard $L_A = \Sigma^\infty_+(A \stackrel{\mathrm{id}}{\to} A)$ as an object of $\mathrm{Mod}_A(\mathsf{DM}(k))$. Let $\phi : A \to B$ be a morphism in $\mathrm{CAlg}(\mathsf{DM}^\otimes(k))$. Let $(-) \otimes_A B : \mathrm{Mod}_A(\mathsf{DM}(k)) \to \mathrm{Mod}_B(\mathsf{DM}(k))$ denote a left adjoint to the forgetful functor $\mathrm{Mod}_B(\mathsf{DM}(k)) \to \mathrm{Mod}_A(\mathsf{DM}(k))$ induced by $A \to B$. Then as in the classical theory of cotangent complexes, there is a canonical morphism $L_A \otimes_A B \to L_B$. Indeed, $L_A \otimes_A B \simeq \Sigma^\infty_+(A \stackrel{\phi}{\to} B)$ when $A \to B$ is thought of as an object of $\mathrm{CAlg}(\mathsf{DM}^\otimes(k))_{/B}$ (see [33, 7.3.2.14, 7.3.3, 7.3.4.18] and Remark 6.7). We define the relative cotangent complex $L_{B/A}$ of $A \to B$ to be a cokernel (cofiber) of $L_A \otimes_A B \to L_B$ in $\mathrm{Mod}_B(\mathsf{DM}(k))$.

Definition 6.1. Let (X, x) be a pointed smooth scheme separated of finite type over k. Let $x^* : M_X \to \mathbf{1}_k = M_{\mathrm{Spec}\, k}$ be the morphism induced by x. We define $LM_{(X,x)}$ to be $L_{M_X} \otimes_{M_X} \mathbf{1}_k$ in $\mathsf{DM}(k)$. Here L_{M_X} belongs to $\mathrm{Mod}_{M_X}(\mathsf{DM}(k))$, and $(-) \otimes_{M_X} \mathbf{1}_k : \mathrm{Mod}_{M_X}(\mathsf{DM}(k)) \to \mathrm{Mod}_{\mathbf{1}_k}(\mathsf{DM}(k)) \simeq \mathsf{DM}(k)$ is induced by x^*. We shall refer to $LM_{(X,x)}$ as the *cotangent motive* of X at x. For $i \in \mathbb{Z}$ and $j \in \mathbb{Z}$, we define $\prod_{i,j}(X, x) := \mathrm{Hom}_{\mathsf{h}(\mathsf{DM}(k))}(LM_{(X,x)}, \mathbf{1}_k(-j)[-i])$.

Remark 6.2. There is a canonical equivalence $LM_{(X,x)} = L_{M_X} \otimes_{M_X} \mathbf{1}_k \simeq L_{\mathbf{1}_k/M_X}[-1]$. Indeed, there is the distinguished triangle (cofiber sequence) arising from $\mathbf{1}_k \to M_X \to \mathbf{1}_k$:

$$L_{M_X} \otimes_{M_X} \mathbf{1}_k \to L_{\mathbf{1}_k/\mathbf{1}_k} \to L_{\mathbf{1}_k/M_X} \to$$

in the homotopy category of $\mathsf{DM}(k)$, see [33, 7.3.3.5]. In addition, $L_{\mathbf{1}_k/\mathbf{1}_k} \simeq 0$. It follows that $L_{M_X} \otimes_{M_X} \mathbf{1}_k \simeq L_{\mathbf{1}_k/M_X}[-1]$.

Remark 6.3. The definition of the cotangent motives makes sense also when we work with an arbitrary coeffiecient ring K of $\mathsf{DM}(k)$.

The main result of this section is the following:

Theorem 6.4. *Let X be a smooth variety over k and let x be a k-rational point. Suppose that k is embedded in $\mathbb{C}$ and the underlying topological space X^t of $X \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, \mathbb{C}$ is simply connected. Then the (singular) realization functor $\mathsf{R} : \mathsf{DM}(k) \to \mathsf{D}(\mathbb{Q})$ carries $LM_{(X,x)}$ to*

$$\bigoplus_{2 \le i} (\pi_i(X^t, x) \otimes_{\mathbb{Z}} \mathbb{Q})^\vee[-i]$$

in $\mathsf{D}(\mathbb{Q}) \simeq \mathrm{Mod}_{\mathbb{Q}}$. *Namely, there is an isomorphism*

$$H^i(\mathsf{R}(LM_{(X,x)})) \simeq (\pi_i(X^t, x) \otimes_{\mathbb{Z}} \mathbb{Q})^\vee$$

for $i \geq 2$*, and* $H^i(\mathsf{R}(LM_{(X,x)})) = 0$ *for* $i < 2$*. Here* $(\pi_i(X^t, x) \otimes_{\mathbb{Z}} \mathbb{Q})^\vee$ *is the dual* $\mathbb{Q}$*-vector space of* $\pi_i(X^t, x) \otimes_{\mathbb{Z}} \mathbb{Q}$*.*

Remark 6.5. Under the hypothesis of Theorem 6.4, the cohomology $H^i(X^t, \mathbb{Q})$ is finite dimensional for any $i \geq 0$. Indeed, the simply connectedness is not necessary for this finiteness. In general, if S is a simply connected topological space whose cohomoogy $H^i(S, \mathbb{Q})$ is finite dimensional for any $i \geq 0$, then $\pi_i(S, s) \otimes_{\mathbb{Z}} \mathbb{Q}$ is finite dimensional for any $i \geq 2$.

6.2 The proof proceeds in several steps.

Lemma 6.6 (Cisinski-Déglise [12]). *Let* $\mathsf{R}_E : \mathsf{DM}^{\otimes}(k) \to \mathsf{D}^{\otimes}(K)$ *be the symmetric monoidal realization functor associated to mixed Weil theory* E *with coefficients in a field* K *of characteristic zero. Let* G *be a right adjoint to* R_E*, that is lax symmeytric monoidal. Let* $G(K)$ *be the commutative algebra object (i.e., an object of* $\mathrm{CAlg}(\mathsf{DM}^{\otimes}(k))$*) where* K *is the unit algebra in* $\mathsf{D}(K)$*. Consider the composition of symmetric monoidal functors*

$$\mathrm{Mod}^{\otimes}_{G(K)}(\mathsf{DM}(k)) \to \mathrm{Mod}^{\otimes}_{\mathsf{R}_E(G(K))}(\mathsf{D}(K)) \to \mathrm{Mod}^{\otimes}_{K}(\mathsf{D}(K)) \simeq \mathsf{D}^{\otimes}(K)$$

where the first arrow is induced by R_E*, and the second arrow is given by the base change* $(-) \otimes_{\mathsf{R}_E(G(K))} K$ *induced by the counit map* $\mathsf{R}_E(G(K)) \to K$*. Then the composite is an equivalence, and* R_E *is equivalent to the base change functor*

$$(-) \otimes_{\mathbf{1}_k} G(K) : \mathsf{DM}^{\otimes}(k) \to \mathrm{Mod}^{\otimes}_{G(K)}(\mathsf{DM}(k)) \simeq \mathsf{D}^{\otimes}(K).$$

Proof. If we verify two conditions

- there is a set $\{M_\lambda\}_{\lambda \in \Lambda}$ of compact and dualizable objects of $\mathsf{DM}(k)$ such that the whole category $\mathsf{DM}(k)$ is the smallest stable subcategory which contains $\{M_\lambda\}_{\lambda \in \Lambda}$ and is closed under small coproducts (that is to say, $\{M_\lambda\}_{\lambda \in \Lambda}$ is a generator of $\mathsf{DM}(k)$),
- each $\mathsf{R}_E(M_\lambda)$ is compact, and there is some $\mu \in I$ such that $\mathsf{R}_E(M_\mu) \simeq K$,

then our assertion follows from [26, Proposition 2.5]. For $X \in \mathrm{Sm}_k$ and $n \in \mathbb{Z}$, $M(X)(n)$ is compact in $\mathsf{DM}(k)$, and the set $\{M(X)(n)\}_{X \in \mathrm{Sm}_k, n \in \mathbb{Z}}$ is a generator of $\mathsf{DM}(k)$. In addition, $M(X)$ is dualizable because it holds if X is projective, and we work with rational coefficients, so that we can use the standard argument based on de Jong's alteration (or one can directly apply a very general result in [13, 4.4.3, 4.4.17]). Since R_E is symmetric monoidal and $M(X)(n)$ is dualizable, $\mathsf{R}_E(M(X)(n))$ is also dualizable. In $\mathsf{D}(K)$, an object is compact if and only if it is dualizable. Finally, $\mathsf{R}_E(M(\mathrm{Spec}\, k)) = \mathsf{R}_E(\mathbf{1}_k) \simeq K$ since R_E is symmetric monoidal. Hence the above two conditions are satisfied. □

According to Lemma 6.6, under the setting of Theorem 6.4, we write $P := G(\mathbb{Q})$ and identify the (singular) realization functor R with $(-) \otimes_{\mathbf{1}_k} P : \mathsf{DM}^{\otimes}(k) \to \mathrm{Mod}^{\otimes}_P(\mathsf{DM}(k)) \simeq \mathsf{D}^{\otimes}(\mathbb{Q})$. The multiplicative realization functor $\mathrm{CAlg}(\mathsf{R}) : \mathrm{CAlg}(\mathsf{DM}^{\otimes}(k)) \to \mathrm{CAlg}_{\mathbb{Q}}$ can naturally be identified with

$$\mathrm{CAlg}(\mathsf{DM}^{\otimes}(k)) \longrightarrow \mathrm{CAlg}(\mathrm{Mod}^{\otimes}_P(\mathsf{DM}(k)) \simeq \mathrm{CAlg}(\mathsf{DM}^{\otimes}(k))_{P/}$$

which sends A to $P \simeq \mathbf{1}_k \otimes P \to A \otimes P$. For the right equivalence, see [33, 3.4.1.7].

We focus on cotangent complexes of commutative dg algebras, that is, objects of $\mathrm{CAlg}_{\mathbb{Q}}$. Let C be an object of $\mathrm{CAlg}_{\mathbb{Q}}$. If we take $\mathcal{C}$ to be $\mathrm{CAlg}_{\mathbb{Q}}$ in the above formalism of cotangent complexes, we have the adjunction

$$\Sigma^{\infty}_{+} : (\mathrm{CAlg}_{\mathbb{Q}})_{/C} \rightleftarrows \mathrm{Sp}((\mathrm{CAlg}_{\mathbb{Q}})_{/C}) \simeq \mathrm{Mod}_C(\mathsf{D}(\mathbb{Q})) : \Omega^{\infty}.$$

Here we abuse notation by using $\Sigma^{\infty}_{+}, \Omega^{\infty}$ again. We define the absolute cotangent complex L_C of C to be $\Sigma^{\infty}_{+}(C \stackrel{\mathrm{id}}{\to} C)$. Given a morphism $C \to D$ we define $L_{D/C}$ to be a cokernel of $D \otimes_C L_C \to L_D$ in $\mathrm{Mod}_D(\mathsf{D}(\mathbb{Q}))$.

Remark 6.7. Let $\mathcal{C}$ be either $\mathrm{CAlg}(\mathsf{DM}^{\otimes}(k))$ or $\mathrm{CAlg}_{\mathbb{Q}} = \mathrm{CAlg}(\mathsf{D}^{\otimes}(\mathbb{Q}))$. More generally, $\mathcal{C}$ could be a presentable ∞-category $\mathrm{CAlg}(\mathcal{D}^{\otimes})$ such that $\mathcal{D}^{\otimes}$ is a symmetric monoidal stable presentable ∞-category whose tensor product $\mathcal{D} \times \mathcal{D} \to \mathcal{D}$ preserves small colimits separately in each variable. Let A and B be objects of $\mathcal{C}$. Let $B \to A$ be a morphism. Consider the adjunction

$$\Sigma^{\infty}_{+} : \mathcal{C}_{/A} \rightleftarrows \mathrm{Sp}(\mathcal{C}_{/A}) \simeq \mathrm{Mod}_A(\mathcal{D}) : \Omega^{\infty}.$$

If we regard $B \to A$ as an object of $\mathcal{C}_{/A}$, then Σ^{∞}_{+} sends $B \to A$ to $L_B \otimes_B A$, where $(-) \otimes_B A : \mathrm{Mod}_B(\mathcal{D}) \to \mathrm{Mod}_A(\mathcal{D})$ denotes the base change functor. It is a direct consequence of a functorial construction of cotangent complexes by using the notion of a tangent bundle in [33, 7.3.2.14] and a presentation of the tangent bundle by a presentable fibration of module categories [33, 7.3.4.18].

Suppose that A is an initial object (that is, a unit algebra). The above adjunction is extended to

$$\mathcal{D} \rightleftarrows \mathcal{C}_{/A} \rightleftarrows \mathrm{Sp}(\mathcal{C}_{/A}) \simeq \mathcal{D}$$

where the left arrow $j : \mathcal{C}_{/A} = \mathrm{CAlg}(\mathcal{D}^{\otimes})_{/A} \to \mathcal{D}$ is the functor which carries $\epsilon : B \to A$ to the kernel (fiber) $\mathrm{Ker}(\epsilon)$ of $B \to A$ in $\mathcal{D}$. The left adjoint $\mathcal{D} \to \mathcal{C}_{/A}$ to j sends $M \in \mathcal{D}$ to $\mathbb{F}_{\mathcal{D}}(M) \to \mathbb{F}_{\mathcal{D}}(0) \simeq A$ determined by $M \to 0$, where $\mathbb{F}_{\mathcal{D}} : \mathcal{D} \to \mathcal{C}$ is the free algebra functor, see Definition 5.1. By the construction of $\mathrm{Sp}(\mathcal{C}_{/A}) \simeq \mathcal{D}$ (cf. [33, 7.3.4.13]), the composite $\mathcal{D} \simeq \mathrm{Sp}(\mathcal{C}_{/A}) \stackrel{\Omega^{\infty}}{\to} \mathcal{C}_{/A} \stackrel{j}{\to} \mathcal{D}$ is naturally equivalent to the identity functor. Thus Σ^{∞}_{+} carries $\mathbb{F}_{\mathcal{D}}(M) \to A$ to M. Namely, $L_{\mathbb{F}_{\mathcal{D}}(M)} \otimes_{\mathbb{F}_{\mathcal{D}}(M)} A \simeq M$.

Remark 6.8. If one considers $x^* : M_X \to \mathbf{1}_k$ to be an object of $\mathrm{CAlg}(\mathsf{DM}(k))_{/\mathbf{1}_k}$, then its image under $\Sigma^{\infty}_{+} : \mathrm{CAlg}(\mathsf{DM}(k))_{/\mathbf{1}_k} \to \mathsf{DM}(k)$ is $LM_{(X,x)}$ (cf. Remark 6.7). The right adjoint $\Omega^{\infty} : \mathsf{DM}(k) \to \mathrm{CAlg}(\mathsf{DM}(k))_{/\mathbf{1}_k}$ sends $LM_{(X,x)}$ to a square zero extension of $\mathbf{1}_k$ by $LM_{(X,x)}$, which is informally given by $\mathbf{1}_k \oplus LM_{(X,x)} \stackrel{\mathrm{pr}_1}{\to} \mathbf{1}_k$, see [33, 7.3.4] for square zero extensions. By the adjunction, we have the unit map $u : M_X \to \mathbf{1}_k \oplus LM_{(X,x)}$ in $\mathrm{CAlg}(\mathsf{DM}(k))_{/\mathbf{1}_k}$. Let $\overline{M}_X$ be the kernel (fiber) of $M_X \to \mathbf{1}_k$ in $\mathsf{DM}(k)$. It gives rise to a morphism in $\mathsf{DM}(k)$

$$h : \overline{M}_X \to LM_{(X,x)}$$

induced by u. This morphism is a motivic version of the dual Hurewicz map.

Lemma 6.9. *Let $\epsilon : A \to \mathbf{1}_k$ be a morphism in $\mathrm{CAlg}(\mathsf{DM}^{\otimes}(k))$, that is, an augmented commutative algebra object in $\mathsf{DM}(k)$. Let $B := \mathsf{R}(A) \to \mathsf{R}(\mathbf{1}_k) = \mathbb{Q}$ be a the image of ϵ in $\mathrm{CAlg}_{\mathbb{Q}}$ under the multiplicative realization functor. Let L_B be the (absolute) cotangent complex of B and let $L_B \otimes_B \mathbb{Q}$ be the base change that lies in $\mathsf{D}(\mathbb{Q})$. Then there is a canonical equivalence*

$$\mathsf{R}(L_A \otimes_A \mathbf{1}_k) \simeq L_B \otimes_B \mathbb{Q}$$

in $\mathsf{D}(\mathbb{Q})$.

Proof. As explained above, Lemma 6.6 allows us to identify the multiplicative realization functor with $\mathrm{CAlg}(\mathsf{DM}^{\otimes}(k)) \to \mathrm{CAlg}(\mathsf{DM}^{\otimes}(k))_{P/} \simeq \mathrm{CAlg}_{\mathbb{Q}}$. Then we have the pushout diagram

$$\begin{array}{ccc} \mathbf{1}_k & \longrightarrow & P \\ \downarrow & & \downarrow \\ A & \longrightarrow & A \otimes P \end{array}$$

in $\mathrm{CAlg}(\mathsf{DM}^{\otimes}(k))$, and B corresponds to the right vertical arrow $P \to A \otimes P$ which we regard as an object of $\mathrm{CAlg}(\mathsf{DM}^{\otimes}(k))_{P/}$. By [33, 7.3.3.8, 7.3.3.15], the absolute cotangent complex of $P \to A \otimes P$ regarded as an object of $\mathrm{CAlg}(\mathsf{DM}^{\otimes}(k))_{P/}$ is equivalent to the relative cotangent complex $L_{A\otimes P/P}$ of the morphism $P \to A \otimes P$ in $\mathrm{CAlg}(\mathsf{DM}(k))$. It follows that $L_{A\otimes P/P} \simeq L_B$ under the canonical equivalence $\mathrm{Mod}_{A\otimes P}(\mathsf{DM}(k)) \simeq \mathrm{Mod}_B(\mathsf{D}(\mathbb{Q}))$. The final equivalence is induced by

$$\begin{aligned} \mathrm{Mod}_{A\otimes P}(\mathsf{DM}(k)) &\simeq \mathrm{Sp}(\mathrm{CAlg}(\mathsf{DM}^{\otimes}(k))_{/A\otimes P}) \\ &\simeq \mathrm{Sp}((\mathrm{CAlg}(\mathsf{DM}^{\otimes}(k))_{P/})_{/A\otimes P}) \\ &\simeq \mathrm{Sp}((\mathrm{CAlg}_{\mathbb{Q}})_{/B}) \\ &\simeq \mathrm{Mod}_B(\mathsf{D}(\mathbb{Q})) \end{aligned}$$

where the first and final equivalences follow from [33, 7.3.4.13], and the second one follows from [33, 7.3.3.9]. Since $A \otimes P \to \mathbf{1}_k \otimes P \simeq P$ corresponds to $B \to \mathbb{Q}$, we see that $L_B \otimes_B \mathbb{Q}$ corresponds to $L_{A\otimes P/P} \otimes_{A\otimes P} P$ in $\mathrm{Mod}_P(\mathsf{DM}(k)) \simeq \mathsf{D}(\mathbb{Q})$. By the base change formula for cotangent complexes [33, 7.3.3.7], $L_{A\otimes P/P} \simeq L_A \otimes_A (A \otimes P)$. Therefore, we obtain

$$L_{A\otimes P/P} \otimes_{A\otimes P} P \simeq L_A \otimes_A (A \otimes P) \otimes_{A\otimes P} P \simeq (L_A \otimes_A \mathbf{1}_k) \otimes P.$$

Note that $\mathsf{R}(L_A \otimes_A \mathbf{1}_k) \simeq (L_A \otimes_A \mathbf{1}_k) \otimes P$ in $\mathrm{Mod}_P(\mathsf{DM}(k))$. Hence our assertion follows. □

The following is a theorem of Sullivan [48, Section 8], reformulated in terms of cotangent complexes.

Lemma 6.10. *Let (S, s) be a simply connected topological space S with a point s. Assume that the cohomology $H^i(S, \mathbb{Q})$ is a finite dimensional $\mathbb{Q}$-vector space for any $i \geq 0$. Let $A_{PL,\infty}(S)$ be the image of $A_{PL}(S)$ in $\mathrm{CAlg}_{\mathbb{Q}}$ (see Section 4). Let $A_{PL,\infty}(S) \to \mathbb{Q}$ be the augmentation induced by s. Then $L_{A_{PL,\infty}(S)} \otimes_{A_{PL,\infty}(S)} \mathbb{Q} \simeq \oplus_{2\leq i}(\pi_i(S, s) \otimes_{\mathbb{Z}} \mathbb{Q})^{\vee}[-i]$ in $\mathsf{D}(\mathbb{Q})$.*

Proof. For ease of notation, we may assume that S is a rational space, so that $\pi_i(S, s)$ is a $\mathbb{Q}$-vector space for each $i \geq 2$. Consider a Postnikov tower

$$S = S_\infty \to \cdots \to S_n \to S_{n-1} \to \cdots \to S_2 \to S_1.$$

We first show our assertion in the case of S_n. The case of $n = 1$ is trivial because S_1 is contractible and $L_{A_{PL,\infty}(S_1)} \simeq 0$. We suppose that our assertion holds for S_{n-1}. Consider the diagram

$$\begin{array}{ccc} K(\pi_n(S, s), n) & \longrightarrow & S_n \\ \downarrow & & \downarrow \\ * & \longrightarrow & S_{n-1} \end{array}$$

where $*$ is a contractible space, $K(\pi_n(S,s),n)$ is an Eilenberg-MacLane space, and we here think of the diagram with a pullback square in $\mathcal{S}$. By a computation for the Eilenberg-MacLane space [16, Section 15 Example 3, Section 12, Example 2], $A_{PL,\infty}(K(\pi_n(S,s),n)) \simeq \mathbb{F}_{\mathbb{Q}}(\pi_n(S,s)^\vee[-n])$ where $\pi_n(S,s)^\vee[-n]$ is the dual $\mathbb{Q}$-vector space placed in cohomological degree n, that we consider to be an object of $\mathsf{D}(\mathbb{Q})$, and $\mathbb{F}_{\mathbb{Q}} : \mathsf{D}(\mathbb{Q}) \to \mathrm{CAlg}_{\mathbb{Q}}$ is the free algebra functor, see Definition 5.1. By [16, Theorem 15.3], $A_{PL,\infty}(K(\pi_n(S,s),n))$ is a pushout of $A_{PL,\infty}(S_n) \leftarrow A_{PL,\infty}(S_{n-1}) \to \mathbb{Q} \simeq A_{PL,\infty}(*)$ in $\mathrm{CAlg}_{\mathbb{Q}}$ (the result found in [16] shows that it is a homotopy pushout in $\mathrm{CAlg}^{dg}_{\mathbb{Q}}$). When S_n and S_{n-1} are equipped with (compatible) base points, $A_{PL,\infty}(K(\pi_n(S,s),n))$ is promoted to a pushout in $(\mathrm{CAlg}_{\mathbb{Q}})_{/\mathbb{Q}}$. Note that $\Sigma^\infty_+ : (\mathrm{CAlg}_{\mathbb{Q}})_{/\mathbb{Q}} \to \mathrm{Sp}((\mathrm{CAlg}_{\mathbb{Q}})_{/\mathbb{Q}}) \simeq \mathsf{D}(\mathbb{Q})$ preserves small colimits. Taking account of Remark 6.7 we have a pushout diagram

$$\begin{array}{ccc} L_{A_{PL,\infty}(S_{n-1})} \otimes_{A_{PL,\infty}(S_{n-1})} \mathbb{Q} & \longrightarrow & L_{A_{PL,\infty}(S_n)} \otimes_{A_{PL,\infty}(S_n)} \mathbb{Q} \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \pi_n(S,s)^\vee[-n] \end{array}$$

in $\mathsf{D}(\mathbb{Q})$. By the assumption, $L_{A_{PL,\infty}(S_{n-1})} \otimes_{A_{PL,\infty}(S_{n-1})} \mathbb{Q} \simeq \oplus_{2\le i\le n-1}\pi_i(S,s)^\vee[-i]$. Then we see that $L_{A_{PL,\infty}(S_n)} \otimes_{A_{PL,\infty}(S_n)} \mathbb{Q}$ is a cokernel (cofiber) of $\pi_n(S,s)^\vee[-n-1] \to \oplus_{2\le i\le n-1}\pi_i(S,s)^\vee[-i]$. Thus the case of S_n follows. Next we show the case of S. For simplicity, $A := A_{PL,\infty}(S)$ and $A_n := A_{PL,\infty}(S_n)$. As the above proof reveals, $L_{A_{n-1}} \otimes_{A_{n-1}} \mathbb{Q} \to L_{A_n} \otimes_{A_n} \mathbb{Q}$ can be identified with the inclusion $\oplus_{2\le i\le n-1}\pi_i(S,s)^\vee[-i] \to \oplus_{2\le i\le n}\pi_i(S,s)^\vee[-i]$. It will suffice to prove that the canonical morphism $\varinjlim_n L_{A_n} \otimes_{A_n} \mathbb{Q} \to L_A \otimes_A \mathbb{Q}$ is an equivalence in $\mathsf{D}(\mathbb{Q})$. Since Σ^∞_+ preserves colimits, it is enough to show that the canoncial morphism $\varinjlim_n A_n \to A$ is an equivalence $\mathrm{CAlg}_{\mathbb{Q}}$. For this we are reduced to proving the canonical map $\varinjlim_n H^i(S_n,\mathbb{Q}) = \varinjlim_n H^i(A_n) \to H^i(S,\mathbb{Q}) = H^i(A)$ is bijective for $i \ge 0$. By applying Serre spectral sequence to the fiber sequence $F_{m,n} = * \times_{S_n} S_m \to S_m \to S_n$ for $n \le m \le \infty$, we see that $H^n(S_n,\mathbb{Q}) \simeq H^n(S_{n+1},\mathbb{Q}) \simeq \ldots \simeq H^n(S,\mathbb{Q})$, so that $\varinjlim_n H^i(S_n,\mathbb{Q}) \simeq H^i(S,\mathbb{Q})$. □

Proof of Theorem 6.4. By Theorem 4.3 and Remark 4.4, the image of $M_X \to \mathbf{1}_k$ can be identified with $A_{PL,\infty}(X^t) \to \mathbb{Q}$ induced by the point x on X^t. Write $B := A_{PL,\infty}(X^t)$. Taking account of Lemma 6.9, we see that $\mathsf{R}(LM_{(X,x)}) \simeq L_B \otimes_B \mathbb{Q}$. Now our assertion follows from Lemma 6.10. □

We would like to relate cotangent motives with partial data of fundamental groups.

Theorem 6.11. *Let (X,x) be a pointed smooth variety over k. Suppose that k is embedded in $\mathbb{C}$. Let $\pi_i(X^t,x)_{uni}$ be the pro-unipotent completion of the fundamental group $\pi_i(X^t,x)$ of X^t over $\mathbb{Q}$. Then the $\mathbb{Q}$-vector space $H^1(\mathsf{R}(LM_{(X,x)}))$ gets identified with the cotangent space of the unipotent affine scheme $\pi_1(X^t,x)_{uni}$ at the origin.*

Proof. As in the proof of Theorem 6.4, the image of $M_X \to \mathbf{1}_k$ can be identified with $A_{PL,\infty}(X^t) \to \mathbb{Q}$ induced by the point x on X^t. Write $B := A_{PL,\infty}(X^t)$. The image of $\mathbf{1}_k \otimes_{M_X} \mathbf{1}_k$ under the multiplicative realization functor can naturally be identified with $\mathbb{Q} \otimes_B \mathbb{Q}$. According to Hochschild-Kostant-Rosenberg (HKR) theorem for $B \in \mathrm{CAlg}_{\mathbb{Q}}$, we have $\mathbb{Q} \otimes_B \mathbb{Q} \simeq \mathbb{Q} \otimes_B B \otimes_{B\otimes B} B \simeq \mathbb{F}_{\mathbb{Q}}((L_B \otimes_B \mathbb{Q})[1])$ (see e.g. [7, Prop. 4.4] for HKR theorem: strictly speaking, the connectivity on B is assumed in loc. cit., but its proof shows that the nonconnective affine case holds). It follows that $H^0(\mathbb{Q} \otimes_B \mathbb{Q}) \simeq H^0(\mathbb{F}_{\mathbb{Q}}((L_B \otimes_B \mathbb{Q})[1]))$ (keep in mind that the dual of the base point $H^0(\mathbb{Q} \otimes_B \mathbb{Q}) \to H^0(\mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{Q}) \simeq \mathbb{Q}$ is identified with $H^0(\mathbb{F}_{\mathbb{Q}}((L_B \otimes_B \mathbb{Q})[1])) \to H^0(\mathbb{F}_{\mathbb{Q}}(0)) \simeq \mathbb{Q}$ induced by $L_B \otimes_B \mathbb{Q} \to L_{\mathbb{Q}} \simeq 0$). Remember that $H^0(\mathbb{Q} \otimes_B \mathbb{Q})$

is isomorphic to the coordinate ring of the pro-unipotent completion $\pi_1(X^t,x)_{uni}$ of $\pi_1(X^t,x)$ over $\mathbb{Q}$, cf. Proposition 7.16. By Lemma 6.9, $\mathsf{R}(LM_{(X,x)}) \simeq L_B \otimes_B \mathbb{Q}$. Let us observe that $H^0(\mathbb{F}_\mathbb{Q}(\mathsf{R}(LM_{(X,x)})[1])) \simeq H^0(\mathbb{F}_\mathbb{Q}((L_B \otimes_B \mathbb{Q})[1]))$ is naturally isomorphic to the free ordinary commutative $\mathbb{Q}$-algebra $\mathbb{F}_{ord}(H^1(L_B \otimes_B \mathbb{Q}))$ generated by the $\mathbb{Q}$-vector space $H^1(L_B \otimes_B \mathbb{Q}) \simeq H^0(\mathsf{R}(LM_{(X,x)})[1])$. Taking account of Lemma 5.19, we are reduced to showing that $H^i(L_B \otimes_B \mathbb{Q}) = 0$ for $i < 1$. Thus, it will suffice to prove the following Lemma. □

Lemma 6.12. *Let B be a commutative dg algebra over $\mathbb{Q}$, which we regard as an object of* $\mathrm{CAlg}_\mathbb{Q}$*. Suppose that we are given an augmentation $B \to \mathbb{Q}$. Assume that $H^0(B) = \mathbb{Q}$, and $H^i(B) = 0$ for $i < 0$. Then $H^i(L_B \otimes_B \mathbb{Q}) = 0$ for $i < 1$.*

Proof. (This fact or equivalent versions is well-known, but we prove it for completeness.) Let $B_0 = \mathbb{Q} \to B_1 \to \cdots \to B_n \to \cdots \to B$ be the inductive sequence associated to the canonical morphism $\mathbb{Q} \to B$ in $\mathrm{CAlg}_\mathbb{Q}$, see Section 5.1.5. By Lemma 5.15, $\varinjlim_n B_n \simeq B$. It follows that $\varinjlim_n L_{B_n} \otimes_{B_n} \mathbb{Q} \simeq L_B \otimes_B \mathbb{Q}$. Therefore, it is enough to show that $H^i(L_{B_n} \otimes_{B_n} \mathbb{Q}) = 0$ for $i < 1$. We will prove, by induction on $n \geq 0$, that (i) $H^0(B_n) = \mathbb{Q}$, $H^i(B_n) = 0$ for $i < 0$, (ii) $H^1(B_n) \to H^1(B)$ is injective, and (iii) $H^i(L_{B_n} \otimes_{B_n} \mathbb{Q}) = 0$ for $i < 1$. For $n = 0$, this is obvious. Assume therefore that all (i), (ii), (iii) hold for n. Let M be the kernel (fiber) of $B_n \to B$ in $\mathsf{D}(\mathbb{Q})$. Then $H^i(M) = 0$ for $i < 2$ by the inductive assumptions (i) and (ii). By definition, $B_{n+1} = B_n \otimes_{\mathbb{F}_\mathbb{Q}(M)} \mathbb{Q}$. By the explicit presentation of the homotopy pushout $B_n \otimes_{\mathbb{F}_\mathbb{Q}(M)} \mathbb{Q}$ (Propsition 5.5 and Remark 5.6), (i) holds for B_{n+1}. In addition, again by the explicit homotopy pushout, we have an exact sequence $0 \to H^1(B_n) \to H^1(B_{n+1}) \to H^2(M)$. Comparing it with the exact sequence $0 \to H^1(B_n) \to H^1(B) \to H^2(M)$ induced by the cofiber sequence $M \to B_n \to B$, we see that $H^1(B_{n+1}) \to H^1(B)$ is injective. Note that $L_{B_{n+1}} \otimes_{B_{n+1}} \mathbb{Q}$ is a cokernel (cofiber) of $L_{\mathbb{F}_\mathbb{Q}(M)} \otimes_{\mathbb{F}_\mathbb{Q}(M)} \mathbb{Q} \to L_{B_n} \otimes_{B_n} \mathbb{Q}$. By Remark 6.7, $L_{\mathbb{F}_\mathbb{Q}(M)} \otimes_{\mathbb{F}_\mathbb{Q}(M)} \mathbb{Q} \simeq M$. Taking account of the inductive assumption (iii) for B_n, we conclude that (iii) holds for B_{n+1}. □

6.3 We use the explicit computations of cohomological motivic algebras in Section 5 to obtain explicit presentations of cotangent motives.

Theorem 6.13. *We have the following explicit presentations:*

(1) Let $\mathbb{P}^n$ be the projective space over a perfect field k and let x be a k-rational point, see Section 5.1.2. Then $LM_{(\mathbb{P}^n,x)} \simeq \mathbf{1}_k(-1)[-2] \oplus \mathbf{1}_k(-n-1)[-2n-1]$.

(2) Let $X = \mathbb{A}^n - \{p\}$ and let x be a k-rational point, see Section 5.1.3. Then

$$LM_{(X,x)} \simeq \mathbf{1}_k(-n)[-2n+1].$$

(3) Let $Y = \mathbb{A}^n - \{p\} - \{q\}$ $(n \geq 2)$ and let y be a k-rational point, see Section 5.1.4. Then

$$LM_{(Y,y)} \simeq \mathbf{1}_k(-n)[-2n+1]^{\oplus 2} \oplus \mathbf{1}_k(-2n)[-4n+3] \oplus \mathbf{1}_k(-3n)[-6n+5]^{\oplus 2} \oplus \ldots.$$

(4) Let G be a semi-abelian variety and let o be the origin, see Section 5.2. Then $LM_{(G,o)} \simeq M_1(G)^\vee$.

Proof. We show (1). We use the notation in Section 5.1.2. By Propsition 5.9,

$$M_{\mathbb{P}^n} \simeq \mathbb{F}(\mathbf{1}_k(-1)[-2]) \otimes_{\mathbb{F}(\mathbf{1}_k(-n-1)[-2n-2])} \mathbf{1}_k.$$

Let $x^*: M_{\mathbb{P}^n} \to \mathbf{1}_k$ be the morphism induced by the k-rational point x. Note that $\mathbb{F}(\mathbf{1}_k(-1)[-2]) \to M_{\mathbb{P}^n} \xrightarrow{x^*} \mathbf{1}_k$ is equivalent to $\mathbb{F}(\mathbf{1}_k(-1)[-2]) \to \mathbb{F}(0) \simeq \mathbf{1}_k$ determined by $\mathbf{1}_k(-1)[-2] \to 0$. Indeed, the morphism $\mathbb{F}(\mathbf{1}_k(-1)[-2]) \to \mathbf{1}_k$ in $\mathrm{CAlg}(\mathsf{DM}^{\otimes}(k))$ is classified by the composite $\mathbf{1}_k(-1)[-2] \hookrightarrow \mathbb{F}(\mathbf{1}_k(-1)[-2]) \to \mathbf{1}_k$ in $\mathsf{DM}(k)$, which is null-homotopic because $\mathrm{CH}^1(\operatorname{Spec} k) = 0$. Similarly, $\mathbb{F}(\mathbf{1}_k(-n-1)[-2n-2]) \to M_{\mathbb{P}^n} \to \mathbf{1}_k$ is equivalent to $\mathbb{F}(\mathbf{1}_k(-n-1)[-2n-2]) \to \mathbb{F}(0)$ determined by $\mathbf{1}_k(-n-1)[-2n-2] \to 0$. Since $\Sigma^\infty_+ : \mathrm{CAlg}(\mathsf{DM}^{\otimes}(k))_{/\mathbf{1}_k} \to \mathrm{Sp}(\mathrm{CAlg}(\mathsf{DM}^{\otimes}(k))_{/\mathbf{1}_k}) \simeq \mathsf{DM}(k)$ preserves small colimits, thus we have the pushout diagram

$$\begin{array}{ccc} L_{\mathbb{F}(\mathbf{1}_k(-n-1)[-2n-2])} \otimes_{\mathbb{F}(\mathbf{1}_k(-n-1)[-2n-2])} \mathbf{1}_k & \longrightarrow & L_{\mathbb{F}(\mathbf{1}_k(-1)[-2])} \otimes_{\mathbb{F}(\mathbf{1}_k(-1)[-2])} \mathbf{1}_k \\ \downarrow & & \downarrow \\ L_{\mathbf{1}_k} & \longrightarrow & LM_{(\mathbb{P}^n, x)} \end{array}$$

in $\mathsf{DM}(k)$, cf. Remark 6.7. Moreover, again by Remark 6.7, the upper left term (resp. the upper right term) is equivalent to $\mathbf{1}_k(-n-1)[-2n-2]$ (resp. $\mathbf{1}_k(-1)[-2]$). The/any morphism $\mathbf{1}_k(-n-1)[-2n-2] \to \mathbf{1}_k(-1)[-2]$ is null-homotopic. Combining this consideration with $L_{\mathbf{1}_k} \simeq 0$, we conclude that $LM_{(\mathbb{P}^n,x)} \simeq \mathbf{1}_k(-1)[-2] \oplus \mathbf{1}_k(-n-1)[-2n-1]$. The cases (2) and (4) are easier than (1) (cf. Proposition 5.12 and Proposition 5.17). We treat the case (3). Recall from Proposition 5.14 that M_Y is equivalent to $\varinjlim_i A_i$ where we can compute A_i in an inductive way: $A_0 = \mathbf{1}_k$, $A_1 = \mathbb{F}(\mathbf{1}_k(-n)[-2n+1]^{\oplus 2})$, $A_2 = A_1 \otimes_{\mathbb{F}(\mathbf{1}_k(-2n)[-4n+2])} \mathbf{1}_k$, $A_3 = A_2 \otimes_{\mathbb{F}(\mathbf{1}_k(-3n)[-6n+4]^{\oplus 2} \oplus \mathbf{1}_k(-4n)[-8n+5])} \mathbf{1}_k$. It will suffice to compute $\varinjlim_i (L_{A_i} \otimes_{A_i} \mathbf{1}_k)$. Using pushout diagrams as above, we have $L_{A_1} \otimes_{A_1} \mathbf{1}_k \simeq \mathbf{1}_k(-n)[-2n+1]^{\oplus 2}$, $L_{A_2} \otimes_{A_2} \mathbf{1}_k \simeq \mathbf{1}_k(-n)[-2n+1]^{\oplus 2} \oplus \mathbf{1}_k(-2n)[-4n+3]$, and $L_{A_3} \otimes_{A_3} \mathbf{1}_k \simeq \mathbf{1}_k(-n)[-2n+1]^{\oplus 2} \oplus \mathbf{1}_k(-2n)[-4n+3] \oplus \mathbf{1}_k(-3n)[-6n+5]^{\oplus 2} \oplus \mathbf{1}_k(-4n)[-8n+6]$. We then find the first few terms of $LM_{(Y,y)}$. □

Remark 6.14. Let us consider a meaning of the presentation of the case of projective spaces. In light of Theorem 6.4, if $k \subset \mathbb{C}$, we have

$$\mathsf{R}(\mathbf{1}_k(-1)) \simeq (\pi_2(\mathbb{CP}^n, x) \otimes_{\mathbb{Z}} \mathbb{Q})^\vee, \quad \mathsf{R}(\mathbf{1}_k(-n-1)) \simeq (\pi_{2n+1}(\mathbb{CP}^n, x) \otimes_{\mathbb{Z}} \mathbb{Q})^\vee.$$

Thus, it is natural to think that $\mathbf{1}_k(1)$ is a motive for $\pi_2(\mathbb{CP}^n, x) \otimes_{\mathbb{Z}} \mathbb{Q}$, and $\mathbf{1}_k(n+1)$ is a motive for $\pi_{2n+1}(\mathbb{CP}^n, x) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Remark 6.15. According to Theorem 6.13 (4), the cotangent motives may also be viewed as a generalization of (the dual of) 1-motives of semi-abelian varieties.

7. Motivic Galois action

Let K be a field of characteristic zero. Let $\mathsf{R}_E : \mathsf{DM}^{\otimes}(k) \to \mathsf{D}^{\otimes}(K)$ be the realization functor associated to a mixed Weil cohomology theory E. In [24] (see also [25], [26]), we constructed a derived affine group scheme MG_E over K out of R_E, which we refer to as the *derived motivic Galois group* with respect to E. It has many favorable properties such as the consistency of motivic conjectures. The most important property of MG_E for us is that it represents the automorphism group of the symmetric monoidal functor R_E, see Definition 7.9 or [24] for the formulation. Moreover, we have the usual affine group scheme MG_E associated to MG_E which we call the *motivic Galois group* with respect to E.

The goal in this section is to construct canonical actions of MG_E on the pro-unipotent completions of fundamental groups and higher homotopy groups, see Theorem 7.17, Corollary 7.18

(in Corollary 7.18, E is singular cohomology theory). It seems difficult to construct them in a direct way. Our strategy can be summarized as follows:

(1) First we will construct an action of MG_E on $\mathrm{CAlg}(\mathsf{R}_E)(M_X)$. Note that a symmetric monoidal natural equivalence from R_E to itself induces a natural equivalence from

$$\mathrm{CAlg}(\mathsf{R}_E) : \mathrm{CAlg}(\mathsf{DM}^{\otimes}(k)) \to \mathrm{CAlg}_K$$

to itself. Actually, there is a canonical morphism from the automorphism group of R_E to the automorphism group of $\mathrm{CAlg}(\mathsf{R}_E)$. Since M_X belongs to $\mathrm{CAlg}(\mathsf{DM}^{\otimes}(k))$ for a smooth scheme X, the automorphism group of $\mathrm{CAlg}(\mathsf{R}_E)$ acts on the image of M_X, e.g. $A_{PL,\infty}(X^t)$ in $\mathrm{CAlg}_{\mathbb{Q}}$. Consequently, it gives rise to an action of MG_E on the image of M_X. We will carry it out in Section 7.1.

(2) In the next step, we focus on the situation of a cosimplicial diagram in $\mathrm{CAlg}(\mathsf{DM}^{\otimes}(k))$ (cf. Section 7.2). The motivating cases come from Section 3.5 and Section 4.5. It yields an action of MG_E on the derived affine group schemes $G_E^{(n)}(X,x)$ in Definition 4.7.

(3) In Sections 7.3 and 7.4, we will show how to obtain actions of MG_E on the pro-unipotent completions of homotopy groups (and related affine group schemes $\overline{G}_E^{(n)}(X,x)$) from those of MG_E on $G_E^{(n)}(X,x)$.

7.1 Our first task is to construct motivic Galois actions on the images of multipilicative realization functors such as $A_{PL}(X^t)$.

7.1.1

Definition 7.1. Let $\mathcal{I}$ be an ∞-category and $D : \mathcal{I} \to \mathrm{Cat}_\infty$ a functor. Suppose that $\mathcal{I}$ has an initial object ξ. Let C be an object of $D(\xi)$. Let $(-)^{\simeq} : \mathrm{Cat}_\infty \to \mathcal{S}$ be the functor which carries an ∞-category $\mathcal{C}$ to its largest Kan subcomplex $\mathcal{C}^{\simeq}$. Namely, it is the right adjoint to the inclusion $\mathcal{S} \to \mathrm{Cat}_\infty$. Let $\mathcal{F}_D \to \mathcal{I}$ be a left fibration obtained by applying the unstraightening functor or relative nerve functor [32] to $\mathcal{I} \to \mathrm{Cat}_\infty \to \mathcal{S}$. By [32, 3.3.3.4], a section $\mathcal{I} \to \mathcal{F}_D$ of $\mathcal{F}_D \to \mathcal{I}$ corresponds to an object in the limit $\varprojlim_{i\in\mathcal{I}} D(i)^{\simeq}$ in $\mathcal{S}$. We let $s : \mathcal{I} \to \mathcal{F}_D$ be the section that corresponds to the image of C under the canonoical functor $D(\xi)^{\simeq} \to \varprojlim_{i\in\mathcal{I}} D(i)^{\simeq}$. Through the correspondence between left fibrations over $\mathcal{I}$ and functors $\mathcal{I} \to \mathcal{S}$ (cf. [32, 3.2, 4.2.4.4]), $\mathcal{F}_D \to \mathcal{I}$ endowed with the section s amounts to the functor $(-)^{\simeq} \circ D : \mathcal{I} \to \mathcal{S}$ with a natural transformation $* \to (-)^{\simeq} \circ D$ from the constant functor $* : \mathcal{I} \to \mathcal{S}$ taking the value Δ^0. By the adjunction, the natural transformation is described as a functor $D_* : \mathcal{I} \to \mathcal{S}_* := \mathcal{S}_{\Delta^0/} \subset \mathrm{Fun}(\Delta^1, \mathcal{S})$ such that the composition $\mathcal{I} \to \mathcal{S}_* \to \mathcal{S}$ with the forgetful functor is $(-)^{\simeq} \circ D$. We shall refer to D_* as the functor extended by C. Let $\mathrm{Grp}(\mathcal{S})$ denote the category of group objects in $\mathcal{S}$ (see e.g. [32, 7.2.2.1], [24, Definition A.2]). Let $\Omega_* : \mathcal{S}_* \to \mathrm{Grp}(\mathcal{S})$ be the functor which carries the based space S to the based loop space $\Omega_*(S)$. We define the automorphism group functor of C over $\mathcal{I}$ to be the composite

$$\mathrm{Aut}_{\mathcal{I}}(C) : \mathcal{I} \xrightarrow{D_*} \mathcal{S}_* \xrightarrow{\Omega_*} \mathrm{Grp}(\mathcal{S}).$$

We usually write $\mathrm{Aut}(C)$ for $\mathrm{Aut}_{\mathcal{I}}(C)$.

Remark 7.2. For any object i in $\mathcal{I}$, the composition $\mathcal{I} \overset{\mathrm{Aut}_{\mathcal{I}}(C)}{\to} \mathrm{Grp}(\mathcal{S}) \to \mathcal{S}$ with the forgetful functor sends i to the ∞-groupoid (space) that is equivalent to the mapping space

$\mathrm{Map}_{D(i)}(f(C), f(C))$ where $f : \xi \to i$ is the canonical functor from the initial object. Indeed, the composite $\mathcal{I} \to \mathcal{S}$ sends i to the fiber product $\Delta^0 \times_{D(i)^{\simeq}} \Delta^0$ in $\mathcal{S}$, defined by the map $\Delta^0 \to D(i)$ determined by $f(C)$. The fiber product $\Delta^0 \times_{D(i)^{\simeq}} \Delta^0$ is explicitly given by the fiber product $\{f(C)\} \times_{D(i)^{\simeq}} \mathrm{Fun}(\Delta^1, D(i)^{\simeq}) \times_{D(i)^{\simeq}} \{f(C)\}$ of (genuine) simplicial sets, that is a model of the mapping space (cf. [32, 1.2.2, 4.2.1.8]).

Replacing the universe $\mathbb{U}$ by a larger universe $\mathbb{U} \in \mathbb{V}$, we define the ∞-category $\widehat{\mathrm{Cat}}_\infty$ of $\mathbb{V}$-small ∞-categories.

Definition 7.3. Let $\mathrm{CAlg}_{(-)} : \mathrm{CAlg}_K \to \widehat{\mathrm{Cat}}_\infty$ be the functor which carries A to CAlg_A where CAlg_A is the ∞-category of commutative ring spectra over A, that is, commutative algebra objects in $\mathrm{Mod}_A^{\otimes}$ (a morphism $A \to A'$ maps to $\mathrm{CAlg}_A \to \mathrm{CAlg}_{A'}$ given by the base change $\otimes_A A'$, see Section 7.1.3 for the formulation). Let C be an object of CAlg_K. We apply Definition 7.1 to $\mathrm{CAlg}_{(-)} : \mathcal{I} = \mathrm{CAlg}_K \to \widehat{\mathrm{Cat}}_\infty$ and C after replacing Cat_∞ and $\mathcal{S}$ by $\widehat{\mathrm{Cat}}_\infty$ and $\widehat{\mathcal{S}}$, respectively. We then define $\mathrm{Aut}_{\mathrm{CAlg}_K}(C) : \mathrm{CAlg}_K \to \mathrm{Grp}(\widehat{\mathcal{S}})$ to be the automorphism group functor of C over CAlg_K.

Let L be an ∞-category. Let $(-)^L : \widehat{\mathrm{Cat}}_\infty \to \widehat{\mathrm{Cat}}_\infty$ be the functor which carries $\mathcal{C}$ to $\mathrm{Fun}(L, \mathcal{C})$. Namely, it is given by cotensoring with L. Let $h : L \to \mathrm{CAlg}_K$ be a functor (which we will consider to be a diagram in CAlg_K indexed by L). Consider the composition

$$\mu_L : \mathrm{CAlg}_K \overset{\mathrm{CAlg}_{(-)}}{\to} \widehat{\mathrm{Cat}}_\infty \overset{(-)^L}{\to} \widehat{\mathrm{Cat}}_\infty.$$

Applying Definition 7.1 to μ_L and h, we define $\mathrm{Aut}_{\mathrm{CAlg}_K}(h) : \mathrm{CAlg}_K \to \mathrm{Grp}(\widehat{\mathcal{S}})$ to be the automorphism group functor of h over CAlg_K. Notice that $\mathrm{Aut}_{\mathrm{CAlg}_K}(C)$ is the special case of $\mathrm{Aut}_{\mathrm{CAlg}_K}(h)$. We usually write $\mathrm{Aut}(C)$ and $\mathrm{Aut}(h)$ for $\mathrm{Aut}_{\mathrm{CAlg}_K}(C)$ and $\mathrm{Aut}_{\mathrm{CAlg}_K}(h)$, respectively.

Definition 7.4. Let $\mathrm{Mod}_{(-)} : \mathrm{CAlg}_K \to \widehat{\mathrm{Cat}}_\infty$ be a functor which carries A to Mod_A (a morphism $A \to A'$ maps to $\mathrm{Mod}_A \to \mathrm{Mod}_{A'}$ given by the base change $\otimes_A A'$, see Section 7.1.3 for the formulation). Let P be an object of $\mathsf{D}(K) \simeq \mathrm{Mod}_K$. Applying Definition 7.1 to $\mathrm{Mod}_{(-)} : \mathrm{CAlg}_K \to \widehat{\mathrm{Cat}}_\infty$ and P, we define $\mathrm{Aut}(P) = \mathrm{Aut}_{\mathrm{CAlg}_K}(P) : \mathrm{CAlg}_K \to \mathrm{Grp}(\widehat{\mathcal{S}})$ to be the automorphism group functor of P over CAlg_K.

Let $\mathsf{R}_E : \mathsf{DM}^{\otimes}(k) \to \mathsf{D}^{\otimes}(K) = \mathrm{Mod}_K^{\otimes}$ be the realization functor associate to a mixed Weil cohomology theory E with coefficients in a field K of characteristic zero. The coefficient field of $\mathsf{DM}(k)$ will be K, but one can also adopt the setting where the coefficient field of $\mathsf{DM}(k)$ is $\mathbb{Q}$ (one may choose either one depending on the purpose). Let $\mathsf{MG}_E = \mathrm{Spec}\, B$ be the derived affine group scheme over K which we call the derived motivic Galois group with respect to E (see [24]). Here the fundamental property of MG_E for us is that it represents the automorphism group functor $\mathrm{Aut}(\mathsf{R}_E) : \mathrm{CAlg}_K \to \mathrm{Grp}(\widehat{\mathcal{S}})$ of the realization functor R_E (see Definition 7.9 for its definition). Namely, if one regards MG_E as a functor $\mathrm{CAlg}_K \to \mathrm{Grp}(\widehat{\mathcal{S}})$, then we have an equivalence $\mathsf{MG}_E \simeq \mathrm{Aut}(\mathsf{R}_E)$.

Proposition 7.5. *Let C be an object of $\mathrm{CAlg}(\mathsf{DM}^{\otimes}(k))$. There is a (canonical) action of MG_E on $\mathrm{CAlg}(\mathsf{R}_E)(C)$. (Recall that $\mathrm{CAlg}(\mathsf{R}_E) : \mathrm{CAlg}(\mathsf{DM}^{\otimes}(k)) \to \mathrm{CAlg}_K$ is the multiplicative realization functor, Section 4.) More precisely, there is a morphism*

$$\mathsf{MG}_E \to \mathrm{Aut}(\mathrm{CAlg}(\mathsf{R}_E)(C))$$

in $\mathrm{Fun}(\mathrm{CAlg}_K, \mathrm{Grp}(\widehat{\mathcal{S}}))$. In particular, we have a (canonical) action of MG_E on $\mathrm{CAlg}(\mathsf{R}_E)(M_X)$. Moreover, the following properties hold:

(1) *The actions are functorial in* $\mathrm{CAlg}(\mathsf{DM}^{\otimes}(k))$*: More precisely, if we let*

$$p : L \to \mathrm{CAlg}(\mathsf{DM}^{\otimes}(k))$$

be a functor from an ∞*-category* L *and let* $h : L \to \mathrm{CAlg}(\mathsf{DM}^{\otimes}(k)) \stackrel{\mathrm{CAlg}(\mathsf{R}_E)}{\to} \mathrm{CAlg}_K$ *be the composition with the multiplicative realization functor, then there is a morphism* $\mathsf{MG}_E \to \mathrm{Aut}(h)$*. For a functor* $g : M \to L$ *of* ∞*-categories, the action (morphism)* $\mathsf{MG}_E \to \mathrm{Aut}(h \circ g)$ *is naturally equivalent to* $\mathsf{MG}_E \to \mathrm{Aut}(h) \to \mathrm{Aut}(h \circ g)$ *where the the first arrow is given by the action on* h*, and the second arrow is induced by the composition with* $M \to L$*.*

(2) *The action is compatible with the formation of colimits: Let* $p : L \to \mathrm{CAlg}(\mathsf{DM}^{\otimes}(k))$ *be a functor from a small* ∞*-category, and* $\overline{p} : L^{\triangleright} \to \mathrm{CAlg}(\mathsf{DM}^{\otimes}(k))$ *a colimit diagram of* p *(here* $(-)^{\triangleright}$ *indicates the right cone [32]). Let* C *be the colimit in* $\mathrm{CAlg}(\mathsf{DM}^{\otimes}(k))$*, that is, the image of the cone point. Let* $q : L \to \mathrm{CAlg}_K$ *and* $\overline{q} : L^{\triangleright} \to \mathrm{CAlg}_K$ *be the composites* $\mathrm{CAlg}(\mathsf{R}_E) \circ p$ *and* $\mathrm{CAlg}(\mathsf{R}_E) \circ \overline{p}$*, respectively. Then the (action) morphism* $\mathsf{MG}_E \to \mathrm{Aut}(\mathrm{CAlg}(\mathsf{R}_E)(C))$ *factors through the morphism* $\mathsf{MG}_E \to \mathrm{Aut}(q)$ *in the sense that the restriction to* L *induces an equivalence* $\mathrm{Aut}(\overline{q}) \stackrel{\sim}{\to} \mathrm{Aut}(q)$*, and the composite*

$$\mathsf{MG}_E \to \mathrm{Aut}(q) \simeq \mathrm{Aut}(\overline{q}) \to \mathrm{Aut}(\mathrm{CAlg}(\mathsf{R}_E)(C))$$

is naturally equivalent to the "action" $\mathsf{MG}_E \to \mathrm{Aut}(\mathrm{CAlg}(\mathsf{R}_E)(C))$*. Here the final arrow is induced by the restriction to the cone point of* $L^{\triangleright}$*.*

(3) *There is a (canonical) action of* MG_E *on* $\mathsf{R}_E(C)$*, that is a morphism* $\mathsf{MG}_E \to \mathrm{Aut}(\mathsf{R}_E(C))$*. We here distinguish the underlying module* $\mathsf{R}_E(C)$ *in* $\mathsf{D}(K)$ *from* $\mathrm{CAlg}(\mathsf{R}_E)(C)$ *in* CAlg_K*. The action on* $\mathrm{CAlg}(\mathsf{R}_E)(C)$ *is compatible with that on* $\mathsf{R}_E(C)$ *in the sense that there is a canonical morphism* $\mathrm{Aut}(\mathrm{CAlg}(\mathsf{R}_E)(C)) \to \mathrm{Aut}(\mathsf{R}_E(C))$ *induced by the forgetful functor, and* $\mathsf{MG}_E \to \mathrm{Aut}(\mathsf{R}_E(C))$ *is equivalent to the composite* $\mathsf{MG}_E \to \mathrm{Aut}(\mathrm{CAlg}(\mathsf{R}_E)(C)) \to \mathrm{Aut}(\mathsf{R}_E(C))$*.*

Corollary 7.6. *Suppose that* k *is embedded in* $\mathbb{C}$*. Let* X^t *be the underlying topological space of* $X \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, \mathbb{C}$*. If* MG *denotes the derived motivic Galois group with respect to the singular cohomology theory, there is a canonical action of* MG *on* $A_{PL,\infty}(X^t) \simeq T_X$*. See the discussion before Proposition 4.1 and Theorem 4.3 for* $A_{PL,\infty}(X^t)$ *and* T_X*.*

Proof. Combine Proposition 7.5 and Theorem 4.3. □

Remark 7.7. Let $A \in \mathrm{CAlg}_K$ and let $g : \Delta^0 \to \mathsf{MG}_E(A)$ be an "A-valued point". Through the equivalence $\mathsf{MG}_E(A) \simeq \mathrm{Aut}(\mathsf{R}_E)(A)$, g may be viewed as an automorphism of the composite $\mathsf{DM}^{\otimes}(k) \stackrel{\mathsf{R}_E}{\to} \mathrm{Mod}_K^{\otimes} \stackrel{\otimes_K A}{\to} \mathrm{Mod}_A^{\otimes}$. It gives rise to an automorphism u of the composite

$$\mathrm{CAlg}(\mathsf{DM}^{\otimes}(k)) \stackrel{\mathrm{CAlg}(\mathsf{R}_E)}{\to} \mathrm{CAlg}_K \stackrel{\otimes_K A}{\to} \mathrm{CAlg}_A$$

(see Section 7.1.2 below). The image $\Delta^0 \to \mathrm{Aut}(\mathrm{CAlg}(\mathsf{R}_E)(C))(A)$ of g under the "action" $\mathsf{MG}_E(A) \to \mathrm{Aut}(\mathrm{CAlg}(\mathsf{R}_E)(C))(A)$ is a class of an equivalence

$$\mathrm{CAlg}(\mathsf{R}_E)(C) \otimes_K A \stackrel{\sim}{\to} \mathrm{CAlg}(\mathsf{R}_E)(C) \otimes_K A$$

in CAlg_A obtained from the automorphism u by evaluating at C (composing with the map $\Delta^0 \to \mathrm{CAlg}(\mathsf{DM}^{\otimes}(k))$ determined by C).

Remark 7.8. One can replace $\mathsf{DM}^{\otimes}(k) = \mathcal{C}^{\otimes}$ by a stable subcategory $\mathcal{E}^{\otimes} \subset \mathsf{DM}^{\otimes}(k)$ that is closed under small colimits and is generated by a small set of dualizable objects. Again by the main result of [24] there is a derived affine group scheme $\mathsf{MG}_{E,\mathcal{E}^{\otimes}}$ that represents $\mathrm{Aut}(\mathsf{R}_E|_{\mathcal{E}^{\otimes}})$, and for $C \in \mathrm{CAlg}(\mathcal{E}^{\otimes})$, $\mathsf{MG}_{E,\mathcal{E}^{\otimes}}$ acts on $\mathrm{CAlg}(\mathsf{R}_E(C))$. In certain good cases, one can obtain $\mathsf{MG}_{E,\mathcal{E}^{\otimes}}$ by means of equivariant bar constructions, see [25], [26], [47].

7.1.2 We start with some ∞-categorical preliminary constructions. To make things elementary, we make some efforts to use the machinery of simplicial categories, i.e., simplicially enriched categories, whereas in the preliminary version of this manuscript in 2016, many constructions heavily rely on the theory of left/(co)Cartesian fibrations.

Let $\mathrm{Cat}_\infty^{\mathrm{sMon},\Delta}$ be a simplicial category defined as follows. The objects of $\mathrm{Cat}_\infty^{\mathrm{sMon},\Delta}$ are symmetric monoidal small ∞-categories $\mathcal{C}^{\otimes} \to \Gamma$. Give two symmetric monoidal ∞-categories $\mathcal{C}^{\otimes} \to \Gamma$ and $\mathcal{D}^{\otimes} \to \Gamma$, we define $\mathrm{Fun}_\Gamma^{\otimes}(\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes})$ to be the full subcategory of $\mathrm{Fun}_\Gamma(\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes})$ that consists of symmetric monoidal functors (cf. [33, 2.1.2]). We define the mapping simplicial set $\mathrm{Map}^{\otimes}(\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes}) := \mathrm{Map}_{\mathrm{Cat}_\infty^{\mathrm{sMon},\Delta}}(\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes})$ to be the largest Kan subcomplex of $\mathrm{Fun}_\Gamma^{\otimes}(\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes})$. The composition is defined by the restriction of composition of function complexes. The ∞-category $\mathrm{Cat}_\infty^{\mathrm{sMon}}$ is defined to be the simplicial nerve of $\mathrm{Cat}_\infty^{\mathrm{sMon},\Delta}$.

We let $\mathrm{Cat}_\infty^{\Delta}$ be the simplicial category defined as follows. Objects are ∞-categories, and given two ∞-categories $\mathcal{C}$ and $\mathcal{D}$, the simplicial set $\mathrm{Map}(\mathcal{C}, \mathcal{D})$ is the largest Kan subcomplex of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$. By definition, the simplicial nerve of $\mathrm{Cat}_\infty^{\Delta}$ is Cat_∞.

Let Kan^{Δ} be the simplicial full subcategory of $\mathrm{Cat}_\infty^{\Delta}$ that consists of Kan complexes. For a symmetric monoidal ∞-category $\mathcal{C}^{\otimes}$, the assignment $\mathcal{D}^{\otimes} \mapsto \mathrm{Map}^{\otimes}(\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes})$ determines a simplicial functor $h^{\Delta}_{\mathcal{C}^{\otimes}} : \mathrm{Cat}_\infty^{\mathrm{sMon},\Delta} \to \mathrm{Kan}^{\Delta}$ in the natural way. Taking the simplicial nerve, we obtain $h_{\mathcal{C}^{\otimes}} := \mathrm{N}(h^{\Delta}_{\mathcal{C}^{\otimes}}) : \mathrm{Cat}_\infty^{\mathrm{sMon}} = \mathrm{N}(\mathrm{Cat}_\infty^{\mathrm{sMon},\Delta}) \to \mathrm{N}(\mathrm{Kan}^{\Delta}) = \mathcal{S}$. We remark that it is equivalent to the functor $\mathrm{Cat}_\infty^{\mathrm{sMon}} \to \mathcal{S}$ corepresented by $\mathcal{C}^{\otimes}$ defined in [32, 5.1.3] (in the dual form). Similarly, for an ∞-category $\mathcal{C}$, the assigment $\mathcal{D} \mapsto \mathrm{Map}(\mathcal{C}, \mathcal{D})$ determines a simplicial functor $h^{\Delta}_{\mathcal{C}} : \mathrm{Cat}_\infty^{\Delta} \to \mathrm{Kan}^{\Delta}$. Taking the simplicial nerve, we obtain $h_{\mathcal{C}} := \mathrm{N}(h^{\Delta}_{\mathcal{C}}) : \mathrm{Cat}_\infty = \mathrm{N}(\mathrm{Cat}_\infty^{\Delta}) \to \mathrm{N}(\mathrm{Kan}^{\Delta}) = \mathcal{S}$.

Next we construct a functor $\mathrm{CAlg} : \mathrm{Cat}_\infty^{\mathrm{sMon}} \to \mathrm{Cat}_\infty$ from the ∞-category of symmetric monoidal (small) ∞-categories to the ∞-category of ∞-categories, which sends $\mathcal{C}^{\otimes}$ to $\mathrm{CAlg}(\mathcal{C}^{\otimes})$. For this purpose we construct a simplicial functor

$$\mathrm{CAlg}^{\Delta} : \mathrm{Cat}_\infty^{\mathrm{sMon},\Delta} \longrightarrow \mathrm{Cat}_\infty^{\Delta}$$

which carries $\mathcal{C}^{\otimes} \to \Gamma$ to $\mathrm{CAlg}(\mathcal{C}^{\otimes}) = \mathrm{Fun}_\Gamma^{\mathrm{lax}}(\Gamma, \mathcal{C}^{\otimes})$ where $\mathrm{Fun}_\Gamma^{\mathrm{lax}}(-,-)$ indicates the full subcategory of $\mathrm{Fun}_\Gamma(-,-)$ that consists of lax symmetric monoidal functors. To do this, given two symmetric monoidal ∞-categories we will define a map of simplicial sets

$$\mathrm{Map}^{\otimes}(\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes}) \to \mathrm{Map}(\mathrm{CAlg}(\mathcal{C}^{\otimes}), \mathrm{CAlg}(\mathcal{D}^{\otimes})).$$

Let K be a simplicial set and $f : K \to \mathrm{Map}^{\otimes}(\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes})$ a map of simplcial sets. The map amounts to a map of marked simplicial sets $\zeta : \mathcal{C}^{\otimes} \times K^{\sharp} \to \mathcal{D}^{\otimes}$ over Γ where $K^{\sharp}$ denotes the marked simplicial sets such that all edges are marked. To the map ζ we associate a map of simplicial sets $\mathrm{CAlg}(\mathcal{C}^{\otimes}) \times K \to \mathrm{CAlg}(\mathcal{D}^{\otimes})$, equivalently $K \to \mathrm{Fun}(\mathrm{CAlg}(\mathcal{C}^{\otimes}), \mathrm{CAlg}(\mathcal{D}^{\otimes}))$ as follows. Note that for a simplicial set S, $S \to \mathrm{Fun}_\Gamma(\Gamma, \mathcal{C}^{\otimes} \times_\Gamma (\Gamma \times K))$ corresponds to a pair of maps $S \times \Gamma \to \mathcal{C}^{\otimes}$ over Γ and $S \times \Gamma \to K$. To $S \to \mathrm{CAlg}(\mathcal{C}^{\otimes}) \times K$ corresponding to $\phi : S \times \Gamma \to \mathcal{C}^{\otimes}$ over Γ and

$\psi : S \to K$ we associate $S \to \mathrm{Fun}_\Gamma(\Gamma, \mathcal{C}^\otimes \times_\Gamma (\Gamma \times K))$ corresponding to the pair $\phi : S \times \Gamma \to \mathcal{C}^\otimes$ over Γ and $S \times \Gamma \stackrel{\mathrm{pr}_1}{\to} S \stackrel{\psi}{\to} K$. It gives rise to a map

$$r : \mathrm{Fun}_\Gamma^{\mathrm{lax}}(\Gamma, \mathcal{C}^\otimes) \times K \to \mathrm{Fun}_\Gamma(\Gamma, \mathcal{C}^\otimes \times K).$$

Let $c : \mathrm{Fun}_\Gamma(\Gamma, \mathcal{C}^\otimes \times K) \times \mathrm{Fun}_\Gamma(\mathcal{C}^\otimes \times K, \mathcal{D}^\otimes) \to \mathrm{Fun}_\Gamma(\Gamma, \mathcal{D}^\otimes)$ be composition. Let $\iota : \Delta^0 \to \mathrm{Fun}_\Gamma(\mathcal{C}^\otimes \times K, \mathcal{D}^\otimes)$ be the map determined by $\mathcal{C}^\otimes \times K \to \mathcal{D}^\otimes$ over Γ that corresponds to f. Consider the following composite

$$\begin{aligned}\mathrm{Fun}_\Gamma^{\mathrm{lax}}(\Gamma, \mathcal{C}^\otimes) \times K \simeq (\mathrm{Fun}_\Gamma^{\mathrm{lax}}(\Gamma, \mathcal{C}^\otimes) \times K) \times \Delta^0 &\stackrel{r \times \iota}{\longrightarrow} \mathrm{Fun}_\Gamma(\Gamma, \mathcal{C}^\otimes \times K) \times \mathrm{Fun}_\Gamma(\mathcal{C}^\otimes \times K, \mathcal{D}^\otimes) \\ &\stackrel{c}{\longrightarrow} \mathrm{Fun}_\Gamma(\Gamma, \mathcal{D}^\otimes).\end{aligned}$$

The image of composition is contained in $\mathrm{Fun}_\Gamma^{\mathrm{lax}}(\Gamma, \mathcal{D}^\otimes)$. Therefore we obtain $\mathrm{CAlg}(\mathcal{C}^\otimes) \times K \to \mathrm{CAlg}(\mathcal{D}^\otimes)$ from f. According to the functoriality with respect to K it yields

$$\mathrm{Map}^\otimes(\mathcal{C}^\otimes, \mathcal{D}^\otimes) \to \mathrm{Fun}(\mathrm{CAlg}(\mathcal{C}^\otimes), \mathrm{CAlg}(\mathcal{D}^\otimes)).$$

Since $\mathrm{Map}^\otimes(\mathcal{C}^\otimes, \mathcal{D}^\otimes)$ is a Kan complex, its image is contained in $\mathrm{Map}(\mathrm{CAlg}(\mathcal{C}^\otimes), \mathrm{CAlg}(\mathcal{D}^\otimes))$. It is straightforward to see that $\mathcal{C}^\otimes \mapsto \mathrm{CAlg}(\mathcal{C}^\otimes)$ and $\mathrm{Map}^\otimes(\mathcal{C}^\otimes, \mathcal{D}^\otimes) \to \mathrm{Map}(\mathrm{CAlg}(\mathcal{C}^\otimes), \mathrm{CAlg}(\mathcal{D}^\otimes))$ determine a simplicial functor $\mathrm{CAlg}^\Delta : \mathrm{Cat}_\infty^{\mathrm{sMon},\Delta} \to \mathrm{Cat}_\infty^\Delta$. Taking the simplicial nerves we obtain a functor of ∞-categories

$$\mathrm{CAlg} : \mathrm{Cat}_\infty^{\mathrm{sMon}} \longrightarrow \mathrm{Cat}_\infty.$$

There is another obvious simplicial functor $\mathrm{For}^\Delta : \mathrm{Cat}_\infty^{\mathrm{sMon},\Delta} \longrightarrow \mathrm{Cat}_\infty^\Delta$ which carries any symmetric monoidal ∞-category $\pi : \mathcal{C}^\otimes \to \Gamma$ to the fiber $\pi^{-1}(\langle 1 \rangle)$, i.e., the underlying ∞-categories $\mathcal{C}$. There is the forgetful functor $\mathrm{CAlg}(\mathcal{C}^\otimes) \to \mathcal{C}$ which is defined as $\mathrm{Fun}_\Gamma^{\mathrm{lax}}(\Gamma, \mathcal{C}^\otimes) \to \mathrm{Fun}_\Gamma(\{\langle 1 \rangle\}, \mathcal{C}^\otimes)$ induced by composition with $\{\langle 1 \rangle\} \to \Gamma$. It gives rise to a simplicial natural transformation $\mathrm{CAlg}^\Delta \to \mathrm{For}^\Delta$.

7.1.3 Recall that $\widehat{\mathrm{Cat}}_\infty$ denotes the ∞-category of $\mathbb{V}$-small ∞-categories. We shall write $\widehat{\mathrm{Cat}}_\infty^{\mathrm{sMon}}$ for the ∞-category of symmetric monoidal $\mathbb{V}$-small ∞-categories. The above construction of $\mathrm{CAlg} : \mathrm{Cat}_\infty^{\mathrm{sMon}} \to \mathrm{Cat}_\infty$ also yields $\widehat{\mathrm{CAlg}} : \widehat{\mathrm{Cat}}_\infty^{\mathrm{sMon}} \to \widehat{\mathrm{Cat}}_\infty$. But for simplicity we write CAlg for $\widehat{\mathrm{CAlg}}$.

Let $\Theta_K : \mathrm{CAlg}_K \to \widehat{\mathrm{Cat}}_\infty^{\mathrm{sMon}}$ be a functor which carries A to $\mathrm{Mod}_A^\otimes$ where $\mathrm{Mod}_A^\otimes$ is the symmetric monoidal ∞-category of A-module spectra (see [33], [24, Appendix A.4] for the precise construction). Any morphism $A \to A'$ maps to the symmetric monoidal functor $\mathrm{Mod}_A^\otimes \to \mathrm{Mod}_{A'}^\otimes$, informally given by the base change $\otimes_A A'$. Let $\mathrm{N}(\mathrm{For}^\Delta) : \widehat{\mathrm{Cat}}_\infty^{\mathrm{sMon}} \to \widehat{\mathrm{Cat}}_\infty$ be the forgetful functor. We define $\mathrm{Mod}_{(-)} : \mathrm{CAlg}_K \to \widehat{\mathrm{Cat}}_\infty$ to be the composite of Θ_K and the forgetful functor. We define $\mathrm{CAlg}_{(-)}$ to be the composite $\mathrm{CAlg}_K \stackrel{\Theta_K}{\longrightarrow} \widehat{\mathrm{Cat}}_\infty^{\mathrm{sMon}} \stackrel{\mathrm{CAlg}}{\longrightarrow} \widehat{\mathrm{Cat}}_\infty$.

Definition 7.9. Let $\rho : \mathrm{CAlg}_K \stackrel{\Theta_K}{\to} \widehat{\mathrm{Cat}}_\infty^{\mathrm{sMon}} \stackrel{h_{\mathsf{DM}(k)^\otimes}}{\longrightarrow} \widehat{\mathcal{S}}$ denote the composite which carries A to $\mathrm{Map}^\otimes(\mathsf{DM}^\otimes(k), \mathrm{Mod}_A^\otimes)$. Let $\mathsf{R}_E : \mathsf{DM}^\otimes(k) \to \mathsf{D}^\otimes(K) = \mathrm{Mod}_K^\otimes$ be the realization functor. It may be viewed as an object of $\mathrm{Map}^\otimes(\mathsf{DM}^\otimes(k), \mathrm{Mod}_K^\otimes)$. Applying Definition 7.1 to $\rho : \mathrm{CAlg}_K \to \widehat{\mathcal{S}}$ and R_E we define the automorphism group functor $\mathrm{Aut}(\mathsf{R}_E) : \mathrm{CAlg}_K \to \mathrm{Grp}(\widehat{\mathcal{S}})$ of R_E over CAlg_K.

Remark 7.10. The definition of $\mathrm{Aut}(\mathsf{R}_E)$ is apparently different from that in [24] because in *loc.cit.* we use the full subcategory $\mathsf{DM}^{\otimes}_{\vee}(k)$ spanned by compact (dualizable) objects instead of $\mathsf{DM}^{\otimes}(k)$. But this point is neglective. Since $\mathsf{DM}^{\otimes}(k)$ is canonically equivalent to the symmetric monoidal ∞-category $\mathrm{Ind}(\mathsf{DM}^{\otimes}_{\vee}(k))$ of Ind-objects, thus by the (symmetric monodial) Kan extension, we see that there is a canonical equivalence $\mathrm{Aut}(\mathsf{R}_E) \simeq \mathrm{Aut}(\mathsf{R}_E|_{\mathsf{DM}^{\otimes}_{\vee}(k)})$ induced by the restriction to $\mathsf{DM}^{\otimes}_{\vee}(k) \subset \mathsf{DM}^{\otimes}(k)$.

7.1.4 Construction of the action/Proof of Proposition 7.5. Let L be an ∞-category. Consider the following three simplicial functors:

- Put $\alpha^{\Delta} = h^{\Delta}_{\mathsf{DM}^{\otimes}(k)} : \widehat{\mathrm{Cat}}_{\infty}^{\mathrm{sMon},\Delta} \to \widehat{\mathrm{Kan}}^{\Delta}$. It sends a symmetric monoidal ∞-category $\mathcal{D}^{\otimes}$ to the Kan complex $\mathrm{Map}^{\otimes}(\mathsf{DM}^{\otimes}(k), \mathcal{D}^{\otimes})$.
- Let $\beta^{\Delta}_L : \widehat{\mathrm{Cat}}_{\infty}^{\mathrm{sMon},\Delta} \to \widehat{\mathrm{Kan}}^{\Delta}$ be a simplicial functor that carries $\mathcal{D}^{\otimes}$ to $\mathrm{Map}(L, \mathrm{CAlg}(\mathcal{D}^{\otimes}))$. It is defined as the composite $\widehat{\mathrm{Cat}}_{\infty}^{\mathrm{sMon},\Delta} \stackrel{\mathrm{CAlg}^{\Delta}}{\longrightarrow} \widehat{\mathrm{Cat}}_{\infty}^{\Delta} \stackrel{h_L}{\longrightarrow} \widehat{\mathrm{Kan}}^{\Delta}$.
- Let $\gamma^{\Delta}_L : \widehat{\mathrm{Cat}}_{\infty}^{\mathrm{sMon},\Delta} \to \widehat{\mathrm{Kan}}^{\Delta}$ be a simplicial functor that carries $\mathcal{D}^{\otimes}$ to $\mathrm{Map}(L, \mathcal{D})$. It is defined as the composite $\widehat{\mathrm{Cat}}_{\infty}^{\mathrm{sMon},\Delta} \stackrel{\mathrm{For}^{\Delta}}{\longrightarrow} \widehat{\mathrm{Cat}}_{\infty}^{\Delta} \stackrel{h_L}{\longrightarrow} \widehat{\mathrm{Kan}}^{\Delta}$.

For each $\mathcal{D}^{\otimes}$, the simplicial functor CAlg^{Δ} induces a map of simplicial sets

$$\mathrm{Map}^{\otimes}(\mathsf{DM}^{\otimes}(k), \mathcal{D}^{\otimes}) \to \mathrm{Map}(\mathrm{CAlg}(\mathsf{DM}^{\otimes}(k)), \mathrm{CAlg}(\mathcal{D}^{\otimes})).$$

It is easy to check that these maps determine a simplicial natural transformation

$$\alpha^{\Delta} \to \beta^{\Delta}_{\mathrm{CAlg}(\mathsf{DM}^{\otimes}(k))}.$$

Similarly, For^{Δ} induces a map of simplicial sets

$$\mathrm{Map}^{\otimes}(\mathsf{DM}^{\otimes}(k), \mathcal{D}^{\otimes}) \to \mathrm{Map}(\mathsf{DM}(k), \mathcal{D}).$$

It gives rise to a simplicial natural transformation $\alpha^{\Delta} \to \gamma^{\Delta}_{\mathsf{DM}(k)}$. Let $L \to \mathrm{CAlg}(\mathsf{DM}^{\otimes}(k))$ be a functor. The composition induces simplicial natural transformation $\beta^{\Delta}_{\mathrm{CAlg}(\mathsf{DM}^{\otimes}(k))} \to \beta^{\Delta}_L$. Also, $L \to \mathrm{CAlg}(\mathsf{DM}^{\otimes}(k)) \stackrel{\mathrm{forget}}{\to} \mathsf{DM}(k)$ induces $\gamma^{\Delta}_{\mathsf{DM}(k)} \to \gamma^{\Delta}_L$.

Now applying the simplicial nerve functor to α^{Δ} we obtain $\alpha = h_{\mathsf{DM}^{\otimes}(k)} : \widehat{\mathrm{Cat}}_{\infty}^{\mathrm{sMon}} \to \widehat{\mathcal{S}}$. Similarly, we obtain $\beta_L, \gamma_L : \widehat{\mathrm{Cat}}_{\infty}^{\mathrm{sMon}} \to \widehat{\mathcal{S}}$ from β^{Δ}_L and γ^{Δ}_L. Consider the simplicial natural transformation $\alpha^{\Delta} \to \beta^{\Delta}_{\mathrm{CAlg}(\mathsf{DM}^{\otimes}(K))} \to \beta^{\Delta}_L$. It determines a natural transformation from α to β_L. In fact, we think of $\alpha^{\Delta} \to \beta^{\Delta}_L$ as a simplicial functor $[1] \times \widehat{\mathrm{Cat}}_{\infty}^{\mathrm{sMon},\Delta} \to \mathrm{Kan}^{\Delta}$ such that $[1] = \{0,1\}$ is the linearly ordered set regarded as a (simplicial) category, and the restriction to $\{0\} \times \widehat{\mathrm{Cat}}_{\infty}^{\mathrm{sMon},\Delta} \to \mathrm{Kan}^{\Delta}$ (resp. $\{1\} \times \widehat{\mathrm{Cat}}_{\infty}^{\mathrm{sMon},\Delta} \to \mathrm{Kan}^{\Delta}$) is α^{Δ} (resp. β^{Δ}_L). Since the simplicial nerve functor preserves products, $\Delta^1 \times \widehat{\mathrm{Cat}}_{\infty}^{\mathrm{sMon}} \simeq \mathrm{N}(\{0 \to 1\} \times \widehat{\mathrm{Cat}}_{\infty}^{\mathrm{sMon},\Delta}) \to \mathrm{N}(\widehat{\mathrm{Kan}}^{\Delta}) = \widehat{\mathcal{S}}$ defines a natural transformation from α to β_L, that is, $\Delta^1 \times \widehat{\mathrm{Cat}}_{\infty}^{\mathrm{sMon}} \to \widehat{\mathcal{S}}$ such that $\{0\} \times \widehat{\mathrm{Cat}}_{\infty}^{\mathrm{sMon}} \to \widehat{\mathcal{S}}$ is α, and $\{1\} \times \widehat{\mathrm{Cat}}_{\infty}^{\mathrm{sMon}} \to \widehat{\mathcal{S}}$ is β_L. Similarly, $\alpha^{\Delta} \to \gamma^{\Delta}_{\mathsf{DM}(k)} \to \gamma^{\Delta}_L$ determines a natural transformation from α to γ_L.

Next for $p : L \to \mathrm{CAlg}(\mathsf{DM}^{\otimes}(k))$ and $h = \mathrm{CAlg}(\mathsf{R}_E) \circ p$, we construct an action of MG_E on $\mathrm{Aut}(h)$ (cf. Definition 7.3). If C is an object of $\mathrm{CAlg}(\mathsf{DM}^{\otimes}(k))$, the automorphism group functor $\mathrm{Aut}(\mathrm{CAlg}(\mathsf{R}_E(C)))$ of $\mathrm{CAlg}(\mathsf{R}_E(C))$ over CAlg_k is nothing but $\mathrm{Aut}(h)$ where $L = \Delta^0$, and the functor $p : \Delta^0 \to \mathrm{CAlg}(\mathsf{DM}^{\otimes}(k))$ is determined by C. Let $\Delta^1 \times \widehat{\mathrm{Cat}}_{\infty}^{\mathrm{sMon}} \to \widehat{\mathcal{S}}$ be the natural transformation from α to β_L defined above. Composing with Θ_K, we have $\Delta^1\times\mathrm{CAlg}_K\to\widehat{\mathcal{S}}$, that is a natural transformation from $\rho=\alpha\circ\Theta_K$ to $(-)^{\simeq}\circ\mu_L=\beta_L\circ\Theta_K$ (we here use the notation in Definition 7.3, 7.9). Remember that R_E is an object of $\mathrm{Map}^{\otimes}(\mathsf{DM}^{\otimes}(k),\mathrm{Mod}_K^{\otimes})$. Thus, as in Definition 7.1, both α and β_L are respectively promoted to functors $\alpha_*,\beta_{L*}:\widehat{\mathrm{Cat}}_\infty^{\mathrm{sMon}}\to\widehat{\mathcal{S}}_*$ extended by R_E and $h\in\mathrm{Map}(L,\mathrm{CAlg}_K)$, and $\Delta^1\times\mathrm{CAlg}_K\to\widehat{\mathcal{S}}$ is promoted to a natural transformation $\Delta^1\times\widehat{\mathrm{Cat}}_\infty^{\mathrm{sMon}}\to\widehat{\mathcal{S}}_*$ from α_* to β_{L*}. Composing $\Omega_*:\widehat{\mathcal{S}}_*\to\mathrm{Grp}(\widehat{\mathcal{S}})$ and Θ_K, we obtain

$$\Delta^1\times\mathrm{CAlg}_K\to\Delta^1\times\widehat{\mathrm{Cat}}_\infty^{\mathrm{sMon}}\to\widehat{\mathcal{S}}_*\to\mathrm{Grp}(\widehat{\mathcal{S}})$$

that is a natural tranformation from $\mathrm{Aut}(\mathsf{R}_E)$ to $\mathrm{Aut}(h)$ (cf. Definition 7.3, 7.9). The equivalence $\mathsf{MG}_E\simeq\mathrm{Aut}(\mathsf{R}_E)$ defines a morphism $\mathsf{MG}_E\simeq\mathrm{Aut}(\mathsf{R}_E)\to\mathrm{Aut}(h)$ in $\mathrm{Fun}(\mathrm{CAlg}_K,\mathrm{Grp}(\widehat{\mathcal{S}}))$. An action of MG_E on h is defined to be this morphism.

We prove the property (1) of Proposition 7.5. For a map $g:M\to L$, there is a simplicial natural transformation $\beta_L^{\Delta}\to\beta_M^{\Delta}$ induced by the composition with g. Therefore, by our construction the functoriality is obvious.

Next we prove the property (2) of Proposition 7.5. Let $\mathcal{K}\hookrightarrow\mathrm{Fun}(L^{\triangleright},\mathrm{CAlg}_A)$ be the full subcategory that consists of those functors $F:L^{\triangleright}\to\mathrm{CAlg}_K$ such that the image of the cone point of $L^{\triangleright}$ is a colimit of the restriction $F|_L$. Then by taking account of left Kan extensions [32, 4.3.2.15] (keep in mind that CAlg_A admits small colimits), the map $\mathrm{Fun}(L^{\triangleright},\mathrm{CAlg}_A)\to\mathrm{Fun}(L,\mathrm{CAlg}_A)$ given by the restriction induces an equivalence $\mathcal{K}\xrightarrow{\sim}\mathrm{Fun}(L,\mathrm{CAlg}_A)$ of ∞-categories. Note that $\overline{p}:L^{\triangleright}\to\mathrm{CAlg}(\mathsf{DM}^{\otimes}(k))$ is a colimit diagram (of $L\to\mathrm{CAlg}(\mathsf{DM}^{\otimes}(k))$). The composite $\overline{q}:L^{\triangleright}\to\mathrm{CAlg}(\mathsf{DM}^{\otimes}(k))\xrightarrow{\mathrm{CAlg}(\mathsf{R}_E)}\mathrm{CAlg}_K$ is also a colimit diagram because $\mathrm{CAlg}(\mathsf{R}_E)$ is a left adjoint. Also, the base change $\otimes_K A:\mathrm{CAlg}_K\to\mathrm{CAlg}_A$ is a left adjoint. Thus, the composite $L^{\triangleright}\xrightarrow{\overline{q}}\mathrm{CAlg}_K\to\mathrm{CAlg}_A$ belongs to $\mathcal{K}$. By these observations, we see that $\mathrm{Aut}(\overline{q})\to\mathrm{Aut}(q)$ induced by the restriction is an equivalence in $\mathrm{Fun}(\mathrm{CAlg}_K,\mathrm{Grp}(\widehat{\mathcal{S}}))$. By the functoriality (1), we have the desired factorization of the action.

Finally, we prove (3). One can define $\mathsf{MG}_E\to\mathrm{Aut}(\mathsf{R}_E(C))$ by using $\alpha^{\Delta}\to\gamma^{\Delta}_{\mathsf{DM}(k)}\to\gamma^{\Delta}_L$ and R_E in the same way as we constructed $\mathsf{MG}_E\to\mathrm{Aut}(\mathrm{CAlg}(\mathsf{R}_E)(C))$ from $\alpha^{\Delta}\to\beta^{\Delta}_{\mathsf{DM}^{\otimes}(k)}\to\beta^{\Delta}_L$ and R_E. There is a simplicial natural transformation $\beta^{\Delta}_L\to\gamma^{\Delta}_L$ which is given by $\mathrm{Map}(L,\mathrm{CAlg}(\mathcal{D}^{\otimes}))\to\mathrm{Map}(L,\mathcal{D})$ induced by the composition with the forgetful functor $\mathrm{CAlg}(\mathcal{D}^{\otimes})\to\mathcal{D}$ for each $\mathcal{D}^{\otimes}$. By the simplicial nerve fucntor and the construction in Definiton 7.1, it gives rise to $\mathrm{Aut}(\mathrm{CAlg}(\mathsf{R}_E)(C))\to\mathrm{Aut}(\mathsf{R}_E(C))$. Note that the simplicial natural transformation $\beta^{\Delta}_L\to\gamma^{\Delta}_L$ commutes with $\alpha^{\Delta}\to\beta^{\Delta}_L$ and $\alpha^{\Delta}\to\gamma^{\Delta}_L$. By this commutativily we see that $\mathsf{MG}_E\to\mathrm{Aut}(\mathrm{CAlg}(\mathsf{R}_E)(C))\to\mathrm{Aut}(\mathsf{R}_E(C))$ is naturally equivalent to $\mathsf{MG}_E\to\mathrm{Aut}(\mathsf{R}_E(C))$. □

Remark 7.11. Let M be an object of $\mathsf{DM}(k)$. Let $\mathbb{F}_{\mathsf{DM}(k)}(M)$ in $\mathrm{CAlg}(\mathsf{DM}^{\otimes}(k))$ be the free commutative algebra object generated by M (see Definition 5.1). Let us observe that the action of MG_E on $\mathrm{CAlg}(\mathsf{R}_E)(\mathbb{F}_{\mathsf{DM}(k)}(M))$ is essentially determined by the action of of MG_E on $\mathsf{R}_E(M)$. Since the realization functor is a left adjoint, there is a canonical equivalence $\mathbb{F}_K(\mathsf{R}_E(M))\simeq\mathrm{CAlg}(\mathsf{R}_E)(\mathbb{F}_{\mathsf{DM}(k)}(M))$ where $\mathbb{F}_K:=\mathbb{F}_{\mathrm{Mod}_K}$ is the free algebra functor $\mathrm{Mod}_K\to\mathrm{CAlg}_K$, i.e., the left adjoint to the forgetful functor. Let S be a space that belongs to $\mathcal{S}$. Let $f:S\to\mathsf{MG}_E(K)\simeq\mathrm{Aut}(\mathsf{R}_E)(K)$ be a morphism (in $\mathcal{S}$). Let

$$\begin{aligned} g:S &\to \mathrm{Aut}(\mathrm{CAlg}(\mathsf{R}_E)\big(\mathbb{F}_{\mathsf{DM}(k)}(M)\big))(K)\\ &\simeq \mathrm{Map}_{\mathrm{CAlg}_K}\big(\mathrm{CAlg}(\mathsf{R}_E)(\mathbb{F}_{\mathsf{DM}(k)}(M)),\mathrm{CAlg}(\mathsf{R}_E)(\mathbb{F}_{\mathsf{DM}(k)}(M))\big)\end{aligned}$$

be a class of the map induced by the action of f. The forgetful functor induces morphisms

$$\begin{aligned}&\mathrm{Map}_{\mathrm{CAlg}_K}\big(\mathrm{CAlg}(\mathsf{R}_E)(\mathbb{F}_{\mathsf{DM}(k)}(M)), \mathrm{CAlg}(\mathsf{R}_E)(\mathbb{F}_{\mathsf{DM}(k)}(M))\big)\\ &\qquad \to \mathrm{Map}_{\mathrm{Mod}_K}\big(\mathrm{CAlg}(\mathsf{R}_E)(\mathbb{F}_{\mathsf{DM}(k)}(M))^\sharp, \mathrm{CAlg}(\mathsf{R}_E)(\mathbb{F}_{\mathsf{DM}(k)}(M))^\sharp\big)\\ &\qquad \simeq \mathrm{Map}_{\mathrm{Mod}_K}\big(\mathbb{F}_K(\mathsf{R}_E(M))^\sharp, \mathbb{F}_K(\mathsf{R}_E(M))^\sharp\big)\end{aligned}$$

in $\mathcal{S}$ where $(-)^\sharp$ here indicates the underlying object. By the compatibility (3) in Proposition 7.5, the image of g is equivalent to the map

$$h : S \to \mathrm{Aut}\big(\mathbb{F}_K(\mathsf{R}_E(M))^\sharp\big)(K) \simeq \mathrm{Map}_{\mathrm{Mod}_K}\big(\mathbb{F}_K(\mathsf{R}_E(M))^\sharp, \mathbb{F}_K(\mathsf{R}_E(M))^\sharp\big)$$

that is determined by the action of f on $\mathbb{F}_K(\mathsf{R}_E(M))^\sharp$. The composition with the canonical (unit) map $\mathsf{R}_E(M) \to \mathbb{F}_K(\mathsf{R}_E(M))^\sharp$ yields the morphisms

$$\begin{aligned}\mathrm{Map}_{\mathrm{Mod}_K}\big(\mathbb{F}_K(\mathsf{R}_E(M))^\sharp, \mathbb{F}_K(\mathsf{R}_E(M))^\sharp\big) &\to \mathrm{Map}_{\mathrm{Mod}_K}\big(\mathsf{R}_E(M), \mathbb{F}_K(\mathsf{R}_E(M))^\sharp\big)\\ &\overset{i}{\leftarrow} \mathrm{Map}_{\mathrm{Mod}_K}\big(\mathsf{R}_E(M), \mathsf{R}_E(M)\big)\end{aligned}$$

in $\mathcal{S}$. Taking account of the functoriality similar to (1) in Proposition 7.5, we see that the image of h in $\mathrm{Map}_{\mathrm{Mod}_K}\big(\mathsf{R}_E(M), \mathbb{F}_K(\mathsf{R}_E(M))^\sharp\big)$ is equivalent to the image of $r : S \to \mathrm{Aut}(\mathsf{R}_E(M))(K) \simeq \mathrm{Map}_{\mathrm{Mod}_K}\big(\mathsf{R}_E(M), \mathsf{R}_E(M)\big)$ that is determined by the action of f on $\mathsf{R}_E(M)$. Note that by the adjunction, the composition gives an equivalence

$$\mathrm{Map}_{\mathrm{CAlg}_K}\big(\mathrm{CAlg}(\mathsf{R}_E)(\mathbb{F}_{\mathsf{DM}(k)}(M)), \mathrm{CAlg}(\mathsf{R}_E)(\mathbb{F}_{\mathsf{DM}(k)}(M))\big) \xrightarrow{\sim} \mathrm{Map}_{\mathrm{Mod}_K}\big(\mathsf{R}_E(M), \mathbb{F}_K(\mathsf{R}_E(M))^\sharp\big)$$

in $\mathcal{S}$. Also, the left arrow i is a fully faithful functor since $\mathsf{R}_E(M) \to \mathbb{F}_K(\mathsf{R}_E(M))^\sharp$ defines a direct summand of $\mathbb{F}_K(\mathsf{R}_E(M))^\sharp$. The image of g in $\mathrm{Map}_{\mathrm{Mod}_K}(\mathsf{R}_E(M), \mathbb{F}_K(\mathsf{R}_E(M))^\sharp)$ lies in the essential image of i. The image of g is equivalent to the image of r under i. One can apply this argument to not only K but arbitrary $A \in \mathrm{CAlg}_K$. We remark that any object of $\mathrm{CAlg}(\mathsf{DM}^\otimes(k))$ is constructed from free commutative algebra objects by forming colimits, see Section 5.1.5.

7.2 Let $\mathrm{Fun}(\mathrm{N}(\Delta^{op}), \mathrm{Aff}_K)$ be the ∞-category of simplicial diagrams in Aff_K. The ∞-category of group objects in Aff_K, i.e., derived affine group schemes, is its full subcategory consisting of those simplicial diagram satisfying the condition of group objects (cf. Section 4.4, see also [24, Appendix]). In Section 7.2 we focus on actions on such objects. We continue to use the notation in Section 7.1.

Let $\mathfrak{C} : \mathrm{N}(\Delta) \to \mathrm{CAlg}(\mathsf{DM}^\otimes(k))$ be a functor which we regard as a cosimplicial diagram of commutative algebra objects in $\mathsf{DM}^\otimes(k)$. Suppose that $\mathfrak{C}^{op} : \mathrm{N}(\Delta)^{op} \to \mathrm{CAlg}(\mathsf{DM}^\otimes(k))^{op}$ is a group object. One of our main examples is the opposite of the group object $\mathcal{G}^{(n+1)}(X,x) : \mathrm{N}(\Delta)^{op} \to \mathrm{CAlg}(\mathsf{DM}^\otimes(k))^{op}$ introduced in Section 3.5. The multiplicative realization functor $\mathrm{CAlg}(\mathsf{R}_E)$ preserves coproducts and sends a unit to $K \in \mathrm{CAlg}_K$. It follows that the composite

$$G_\bullet : \mathrm{N}(\Delta)^{op} \overset{\mathfrak{C}^{op}}{\to} \mathrm{CAlg}(\mathsf{DM}^\otimes(k))^{op} \overset{\mathrm{CAlg}(\mathsf{R}_E)}{\longrightarrow} \mathrm{CAlg}_K^{op} = \mathrm{Aff}_K$$

is a group object, that is, a derived affine group scheme over K. We denote it simply by G.

Invoking Proposition 7.5 (see also Definition 7.3) to the opposite of the group object $G_\bullet^{op} = h : \mathrm{N}(\Delta) = L \to \mathrm{CAlg}_K$, we get a morphism

$$\mathsf{MG}_E \to \mathrm{Aut}(G_\bullet^{op})$$

in $\mathrm{Fun}(\mathrm{CAlg}_K, \mathrm{Grp}(\widehat{\mathcal{S}}))$, that is, an action of MG_E on $G_\bullet^{op}$. Put $\mathrm{Aut}(G) := \mathrm{Aut}(G_\bullet^{op})$. Thus we have

Proposition 7.12. *Let $\mathfrak{C}^{op} : \mathrm{N}(\Delta)^{op} \to \mathrm{CAlg}(\mathsf{DM}^{\otimes}(k))^{op}$ be a group object. Let G be the derived affine group scheme over K that is induced by $\mathfrak{C}^{op}$. Then there is a (canonical) action of MG_E on G, that is a morphism $\mathsf{MG}_E \to \mathrm{Aut}(G)$ in $\mathrm{Fun}(\mathrm{CAlg}_K, \mathrm{Grp}(\widehat{\mathcal{S}}))$.*

Remark 7.13. We remark that informally $\mathrm{Aut}(G_\bullet^{op})$ is the automorphism group of the cosimplicial object $G_\bullet^{op}$ in CAlg_K. Therefore, by our convention $\mathrm{Aut}(G) = \mathrm{Aut}(G_\bullet^{op})$, the morphism $\mathsf{MG}_E \to \mathrm{Aut}(G)$ should be viewed as the "right action" on G that corresponds to the "left action" on $G_\bullet^{op}$.

Remark 7.14. The action is functorial with repect to a morphism of derived affine group schemes. Let $\mathfrak{C}'^{op} : \mathrm{N}(\Delta)^{op} \to \mathrm{CAlg}(\mathsf{DM}^{\otimes}(k))^{op}$ be another group object and $G'_\bullet : \mathrm{N}(\Delta)^{op} \to \mathrm{CAlg}_K^{op} = \mathrm{Aff}_K$ the derived affine group scheme induced by the composition with the multiplicative realization functor. Suppose that there is a morphism (i.e., a natural transformation) $\mathfrak{C}^{op} \to \mathfrak{C}'^{op}$. It gives rise to $\theta : \Delta^1 \times \mathrm{N}(\Delta) \to \mathrm{CAlg}(\mathsf{DM}^{\otimes}(k)) \to \mathrm{CAlg}_K$, such that $\{0\} \times \mathrm{N}(\Delta) \to \mathrm{CAlg}_K$ is $G_\bullet'^{op}$, and $\{0\} \times \mathrm{N}(\Delta) \to \mathrm{CAlg}_K$ is $G_\bullet^{op}$. By (1) of Proposition 7.5 the actions of MG_E on $G_\bullet^{op}$ and $G_\bullet'^{op}$ are simultaneously promoted to an action on $\mathrm{Aut}(\theta)$, i.e., $\mathsf{MG}_E \to \mathrm{Aut}(\theta)$.

Example 7.15. Let $(X, x : \mathrm{Spec}\, k \to X)$ be a pointed smooth variety over k. As discussed in Section 4.5 it gives rise to a derived affine group scheme $G_E^{(n)}(X, x) : \mathrm{N}(\Delta)^{op} \to \mathrm{Aff}_K$. Therefore, MG_E acts on $G_E^{(n)}(X, x)$.

7.3 In [24] we defined the motivic Galois group MG_E of $\mathsf{DM}(k)$ (with respect to E) to be a usual affine group scheme over K (i.e., a pro-algebraic group) obtained from MG_E. Also, we can construct a usual affine group scheme $\overline{G}_E^{(n)}(X, x)$ from $G_E^{(n)}(X, x)$, Example 7.15. In general, if G is a derived affine group scheme over the field of characteristic zero K, one can obtain a usual affine group scheme (i.e., pro-algebraic group) $\overline{G}$ over K from G, which we will call the underlying affine group scheme (cf. [24]). We briefly review the procedure.

Let CAlg_K^{dg} be the category of commutative dg algebras C over K (cf. Section 2). Let $\mathrm{CAlg}_K^{dg,\geq 0}$ be the full subcategory of CAlg_K^{dg} that consists of those objects C such that $H^i(C) = 0$ for $i < 0$. It admits a combinatorial model category structure such that a morphism $f : C \to C'$ is a weak equivalence (resp. fibration) if the underlying map is a quasi-isomorphism (resp. a surjective in each degree), see [41, Proposition 5.3] or [17, Theorem 6.2.6]. Any object is fibrant. Any ordinary commutative algebra over K is a cofibrant object in $\mathrm{CAlg}_K^{dg,\geq 0}$ when it is regarded as a commutative dg algebra placed in degree zero. The inclusion $\mathrm{CAlg}_K^{dg,\geq 0} \hookrightarrow \mathrm{CAlg}_K^{dg}$ is a right Quillen functor. Its left adjoint $\tau : \mathrm{CAlg}_K^{dg} \to \mathrm{CAlg}_K^{dg,\geq 0}$ carries C to the quotient of C by the differential graded ideal generated by elements $x \in C^i$ for $i < 0$. Namely, we have a Quillen adjunction $\tau : \mathrm{CAlg}_K^{dg} \rightleftarrows \mathrm{CAlg}_K^{dg,\geq 0}$. We shall write $\mathrm{CAlg}_K^{\geq 0}$ for the ∞-category obtained from the full subcatgory of cofibrant objects in $\mathrm{CAlg}_K^{dg,\geq 0}$ by inverting weak equivalences. The Quillen adjunction induces an adjunction of ∞-categories

$$\tau : \mathrm{CAlg}_K \rightleftarrows \mathrm{CAlg}_K^{\geq 0}$$

[36] where by ease of notation we write τ also for the induced left adjoint functor $\mathrm{CAlg}_K \to \mathrm{CAlg}_K^{\geq 0}$. We let $G = \mathrm{Spec}\, C$ be a derived affine group scheme over K such that $C \in \mathrm{CAlg}_K$. The functor τ preserves colimits, especially coproducts. We put $\mathrm{Aff}_K^{\geq 0} = (\mathrm{CAlg}_K^{\geq 0})^{op}$. We write $\mathrm{Spec}\, R$ for the object in $\mathrm{Aff}_K^{\geq 0}$ corresponding to $R \in \mathrm{CAlg}_K^{\geq 0}$. Then $\mathrm{Spec}\, \tau C$ inherits

a group structure from $G = \operatorname{Spec} C$. Namely, $\operatorname{Spec} \tau C$ is a group object in $\mathrm{Aff}_K^{\geq 0}$. There is a fully faithful left adjoint $\mathrm{CAlg}_K^{\mathrm{dis}} \to \mathrm{CAlg}_K^{\geq 0}$ induced by the natural inclusion from the category of ordinary commutative K-algebras to $\mathrm{CAlg}_K^{dg,\geq 0}$. Its right adjoint $\mathrm{CAlg}_K^{\geq 0} \to \mathrm{CAlg}_K^{\mathrm{dis}}$ is given by taking the cohomology $C \mapsto H^0(C)$. The inclusion $\mathrm{CAlg}_K^{\mathrm{dis}} \to \mathrm{CAlg}_K$ is canonically equivalent to the composite $\mathrm{CAlg}_K^{\mathrm{dis}} \to \mathrm{CAlg}_K^{\geq 0} \to \mathrm{CAlg}_K$. Also, the left adjoint τ is compatible with inclusions $\mathrm{CAlg}_K^{\mathrm{dis}} \subset \mathrm{CAlg}_K$ and $\mathrm{CAlg}_K^{\mathrm{dis}} \subset \mathrm{CAlg}_K^{\geq 0}$ (use the fact that any object C in $\mathrm{CAlg}_K^{\geq 0}$ is the limit of a cosimplicial diagram of ordinary K-algebras). Interpret G as a functor $\mathrm{CAlg}_K \to \mathrm{Grp}(\mathcal{S})$. Its restriction $G^\circ := G|_{\mathrm{CAlg}_K^{\mathrm{dis}}} : \mathrm{CAlg}_K^{\mathrm{dis}} \to \mathrm{Grp}(\mathcal{S})$ is naturally equivalent to the functor given by $A \mapsto \mathrm{Map}_{\mathrm{CAlg}_K^{\geq 0}}(\tau C, A)$. The structure of a commutative Hopf ring spectrum on τC over K (that is, the "dual" of the group structure on $\operatorname{Spec} \tau C$ in $\mathrm{Aff}_K^{\geq 0}$, see [24, Appendix]) gives the structure of a commutative Hopf ring on $H^0(\tau C)$ over K. Namely, the comultiplication $\tau C \to \tau C \otimes_K \tau C$, the counit $\tau C \to K$ and the antipode give rise to the structure of comultiplication $H^0(\tau C) \to H^0(\tau C \otimes_K \tau C) \simeq H^0(\tau C) \otimes_K H^0(\tau C)$ of $H^0(\tau C)$, etc. We denote the associated affine group scheme by $\overline{G} = \operatorname{Spec} H^0(\tau C)$. We shall refer to $\overline{G}$ as the *underlying affine group scheme* of G (or the *coarse moduli space* for G as in [24]). The assignment $G \mapsto \overline{G}$ is functorial and we actually have a functor $\mathrm{Grp}(\mathrm{Aff}_K) \to \mathrm{Grp}(\mathrm{Aff}_K^{\mathrm{dis}})$ which sends G to the associated affine group scheme $\overline{G}$. By the adjunction, the natural morphism $\pi : \operatorname{Spec} \tau C \to \overline{G} = \operatorname{Spec} H^0(\tau C)$ is universal among morphisms to ordinary affine schemes over K in $\mathrm{h}(\mathrm{Aff}_K^{\geq 0})$ (note that $\mathrm{Aff}_K^{\geq 0}$ contains $\mathrm{Aff}_K^{\mathrm{dis}}$ as a full subcategory). Namely, if $\phi : \operatorname{Spec} \tau C \to H$ is a morphism to an ordinary affine scheme H in $\mathrm{h}(\mathrm{Aff}_K^{\geq 0})$, there is a unique morphism $\psi : \overline{G} \to H$ such that $\phi = \psi \circ \pi$. In addition, H is an affine group scheme and $\phi : \operatorname{Spec} \tau C \to H$ is a homomorphism to the affine group scheme over K, then there is a unique homomorphism $\psi : \overline{G} \to H$ in $\mathrm{h}(\mathrm{Grp}(\mathrm{Aff}_K^{\geq 0})$ such that $\phi = \psi \circ \pi$.

As mentioned above, we define MG_E to be the underlying affine group scheme of MG_E. For the properties of MG_E we refer to [24], [25], [26], [27].

We define $\overline{G}^{(n)}(X, x) := \overline{G}_E^{(n)}(X, x)$ to be the underlying affine group scheme of $G_E^{(n)}(X, x)$ (cf. Section 4.5).

We consider a geometric interpretation of $\overline{G}^{(n)}(X, x)$. Suppose that $K = \mathbb{Q}$ and the base field k is embeded in $\mathbb{C}$. We consider the case when the realization functor is associated to singular cohomology theory.

Proposition 7.16. *Let $(X, x : \operatorname{Spec} k \to X)$ be a pointed smooth variety over k. Let $\pi_i(X^t, x)$ be the homotopy group of the underlying topological space $X^t = X \times_{\operatorname{Spec} k} \operatorname{Spec} \mathbb{C}$. For any $n \geq 1$, the affine group schemes $\overline{G}^{(n)}(X, x)$ is a unipotent group scheme (i.e., a pro-unipotent algebraic group). Moreover, $\overline{G}^{(1)}(X, x)$ is the pro-unipotent completion of $\pi_1(X^t, x)$ over $K = \mathbb{Q}$. Suppose further that the topological space X^t is nilpotent and of finite type (e.g. simply connected smooth varieties). Then $\overline{G}^{(n)}(X, x)$ is a pro-unipotent completion of $\pi_n(X^t, x)$ for $n \geq 2$.*

Before proceeding to the proof, we briefly recall the notion of affinization (affination in French) studied in [49] (in [49], cosimplicial algebras are used instead of dg algebras, see [17, 6.4] for the comparison as a Quillen equivalence between the model category of cosimplicial algebras and $\mathrm{CAlg}_K^{dg,\geq 0}$). Let $\widehat{\mathrm{CAlg}}_K^{\geq 0}$ be the $\mathbb{V}$-version of $\mathrm{CAlg}_K^{\geq 0}$ (cf. Section 7.3). Write $\widehat{\mathrm{Aff}}_K^{\geq 0} := (\widehat{\mathrm{CAlg}}_K^{\geq 0})^{op}$. We write $\operatorname{Spec} R$ for an object of $\widehat{\mathrm{Aff}}_K^{\geq 0}$ corresponding to $R \in \widehat{\mathrm{CAlg}}_K^{\geq 0}$. There is an adjunction

$$\mathcal{O} : \mathrm{Fun}(\mathrm{CAlg}_K^{\mathrm{dis}}, \widehat{\mathcal{S}}) \rightleftarrows \widehat{\mathrm{Aff}}_K^{\geq 0}$$

where $\mathcal{O}$ is a left Kan extension of the inclusion $\mathrm{Aff}_K^{\mathrm{dis}} \hookrightarrow \widehat{\mathrm{Aff}}_K^{\geq 0}$ along the Yoneda embedding $\mathrm{Aff}_K^{\mathrm{dis}} \to \mathrm{Fun}(\mathrm{CAlg}_K^{\mathrm{dis}}, \widehat{\mathcal{S}})$ (cf. [49, Section 2.2]). The right adjoint sends $R \in \mathrm{CAlg}_K^{\geq 0}$ to the functor $h_R : \mathrm{CAlg}_K^{\mathrm{dis}} \to \widehat{\mathcal{S}}$ informally given by $A \mapsto \mathrm{Map}_{\widehat{\mathrm{CAlg}_K^{\geq 0}}}(R, A)$. The restriction $\mathrm{Aff}_K^{\geq 0} = (\mathrm{CAlg}_K^{\geq 0})^{op} \to \mathrm{Fun}(\mathrm{CAlg}_K^{\mathrm{dis}}, \widehat{\mathcal{S}})$ of the right adjoint is fully faithful. Let $F : \mathrm{CAlg}_K^{\mathrm{dis}} \to \widehat{\mathcal{S}}$ be a functor. If $\mathcal{O}(F)$ belongs to $\mathrm{Aff}_K^{\geq 0}$ (not to $\widehat{\mathrm{Aff}}_K^{\geq 0}$), we refer to $\mathcal{O}(F)$ as the affinization of F. An object P in $\mathcal{S}$ can be viewed as the constant functor $\mathrm{CAlg}_K^{\mathrm{dis}} \to \widehat{\mathcal{S}}$ with value P. One can consider the affinization of the space $P \in \mathcal{S}$. The composite $\mathcal{S} = \mathrm{Fun}(\Delta^0, \mathcal{S}) \to \mathrm{Fun}(\mathrm{CAlg}_K^{\mathrm{dis}}, \widehat{\mathcal{S}}) \to \widehat{\mathrm{Aff}}_K^{\geq 0}$ preserves small colimits and sends a contractible space to $\mathrm{Spec}\, K$, where the first arrow is the functor given by the composition with $\mathrm{CAlg}_K^{\mathrm{dis}} \to \Delta^0$. Consequently, the composite carries the space $S \in \mathcal{S}$ to $\mathrm{Spec}\, K^S$ where K^S is the cotensor with the space S. By Proposition 4.1 and Remark 4.2, we conclude that $\mathcal{S} \to \mathrm{Aff}_K^{\geq 0} \to \mathrm{Aff}_K$ is equivalent to $A_{PL,\infty}$. By Theorem 4.3, $\mathrm{Spec}\, T_X$ in Aff_K is the affinization of X^t.

Proof. There are several ways to prove the assertion, and we will give one of them. We treat the case $n = 1$. Let $G^{(1)}(X,x)^\circ : \mathrm{CAlg}_K^{\mathrm{dis}} \subset \mathrm{CAlg}_K \to \mathrm{Grp}(\widehat{\mathcal{S}})$ denote the restriction. It carries A to $\Omega_* \mathrm{Spec}\, T_X(A)$, where $\mathrm{Spec}\, T_X(A)$ is the space of A-valued points on $\mathrm{Spec}\, T_X$, and $\Omega_* \mathrm{Spec}\, T_X(A)$ is its base loop space (the base point comes from $x_A : \mathrm{Spec}\, A \to \mathrm{Spec}\, K \to \mathrm{Spec}\, T_X$). We let $G_\circ^{(1)}(X,x) : \mathrm{CAlg}_K^{\mathrm{dis}} \to \mathrm{Grp}(\mathrm{Set})$ be the sheaf of groups with respect to fpqc topology associated to the presheaf $A \mapsto \pi_0(\Omega_* \mathrm{Spec}\, T_X(A)) \simeq \pi_1(\mathrm{Spec}\, T_X(A), x_A)$. Then according to [49, 2.4.5] (or [34, 4.4.8]), $G_\circ^{(1)}(X,x)$ is represented by a unipotent affine group scheme (i.e., a pro-unipotent algebraic group). (We remark that there is a canonical equivalence $\mathrm{Map}_{\mathrm{CAlg}_K}(T_X, A) \simeq \mathrm{Map}_{\mathrm{CAlg}_K^{\geq 0}}(K^{X^t}, A)$ for any $A \in \mathrm{CAlg}_K^{\mathrm{dis}}$, see [41, 7.2].) Note that the natural morphism $G^{(1)}(X,x)^\circ \to G_\circ^{(1)}(X,x)$ is universal among morphisms to sheaves of groups on $\mathrm{CAlg}_K^{\mathrm{dis}}$. On the other hand, there is the natural map $G^{(1)}(X,x)^\circ \to \overline{G}^{(1)}(X,x)$ (recall that if $G^{(1)}(X,x) = \mathrm{Spec}\, C$, the restriction $G^{(1)}(X,x)^\circ$ is represented by $\mathrm{Spec}\, \tau C$). Consequently, by the universal property there is a natural morphism $G_\circ^{(1)}(X,x) \to \overline{G}^{(1)}(X,x)$ of affine group schemes over K. We wish to show that it is an isomorphism. Since $K = \mathbb{Q}$ is characteristic zero and $G_\circ^{(1)}(X,x) \to \overline{G}^{(1)}(X,x)$ is a morphism as affine group schemes over K, it is enough to prove that for any algebraically closed field L, the induced map $G_\circ^{(1)}(X,x)(L) \to \overline{G}^{(1)}(X,x)(L)$ of sets of L-valued points is bijective. In fact, according to [24, Theorem 5.17] (its proof that works also for $G^{(1)}(X,x)$ instead of MG_E) and [34, VIII 4.4.8], we see that $G_\circ^{(1)}(X,x)(L) \to \overline{G}^{(1)}(X,x)(L)$ is bijective. It follows that $\overline{G}^{(1)}(X,x)$ is a unipotent affine group scheme. By [49, 2.4.11] and Theorem 4.3, the group scheme $G_\circ^{(1)}(X,x) \simeq \overline{G}^{(1)}(X,x)$ is naturally isomorphic to a pro-unipotent completion of $\pi_1(X^t,x)$ (that is endowed with the morphism form the constant functor with value $\pi_1(X^t,x)$). The case of $n \geq 2$ is similar. If $G^{(n)}(X,x)^\circ : \mathrm{CAlg}_K^{\mathrm{dis}} \subset \mathrm{CAlg}_K \to \mathrm{Grp}(\widehat{\mathcal{S}})$ denotes the restriction of $G^{(n)}(X,x)$, it carries A to the n-fold loop space $\Omega_*^n \mathrm{Spec}\, T_X(A)$. As in the case of $n = 1$ ([49, 2.4.5]), we observe that the sheaf associated to the presheaf $A \mapsto \pi_n(\mathrm{Spec}\, T_X(A), x_A)$ is isomorphic to $\overline{G}^{(n)}(X,x)$. Then the final assertion follows from [49, 2.5.3]. □

7.4 We will construct an action of the motivic Galois group MG_E on the group scheme $\overline{G}^{(n)}(X,x) := \overline{G}_E^{(n)}(X,x)$. Unfortunately, if one does not assume motivic conjectures that imply the existence of a motivic t-structure, it seems difficult to obtain an action of MG_E on $\overline{G}^{(n)}(X,x)$ from that of MG_E on $G^{(n)}(X,x)$ in a purely categorical way. To overcome this issue, we use a method of homological algebras, which yields a natural action of MG_E on $\overline{G}_E^{(n)}(X,x)$.

For a usual affine group scheme H over K, we let $\Gamma(H)$ be the (ordinary) coordinate ring on H, that is a commutative Hopf ring over K. We let $\mathrm{Aut}(H):\mathrm{CAlg}_K^{\mathrm{dis}}\to\mathrm{Grp}(\mathrm{Set})$ be the functor which assigns A to the group of automorphisms of the commutative Hopf ring $\Gamma(H)\otimes_K A\stackrel{\sim}{\to}\Gamma(H)\otimes_K A$ over A.

Theorem 7.17. *Let (X,x) be a pointed smooth variety over k. Then there is a (canonical) morphism $MG_E\to\mathrm{Aut}(\overline{G}^{(n)}(X,x))$ in $\mathrm{Fun}(\mathrm{CAlg}_K^{\mathrm{dis}},\mathrm{Grp}(\mathrm{Set}))$, that is, an action of MG_E on $\overline{G}^{(n)}(X,x)$. In other words, the action is described as an action on the scheme $\overline{G}^{(n)}(X,x)$*

$$\overline{G}^{(n)}(X,x)\times MG_E\to\overline{G}^{(n)}(X,x)$$

which is compatible with the group structure. Moreover, the following properties hold:

(1) The action is functorial: Let $\phi:(X,x)\to(Y,y)$ be a morphism of smooth varieties over k that sends x to y. Let $\phi_*:\overline{G}^{(n)}(X,x)\to\overline{G}^{(n)}(Y,y)$ be the induced morphism of group schemes. Then the action of MG_E commutes with ϕ_*.

(2) The action has a moduli theoretic interpretation in a coarse sense (see Remark 7.19).

Corollary 7.18. *Suppose that k is embedded in $\mathbb{C}$ and consider the case of singular realization. Let $\pi_i(X^t,x)_{uni}$ be the pro-unipotent completion of $\pi_i(X^t,x)$ over $\mathbb{Q}$. Then we have a canonical action*

$$\pi_1(X^t,x)_{uni}\times MG\to\pi_1(X^t,x)_{uni}.$$

If X^t is nilpotent and of finite type, there is a canonical action of MG on $\pi_n(X^t,x)_{uni}$ for $n\geq 2$.

Proof. It follows from Theorem 7.17 and Proposition 7.16. □

Construction of an action/Proof of Theorem 7.17. Let $G^{(n)}(X,x):\mathrm{N}(\Delta)^{op}\to\mathrm{Aff}_K$ be the derived affine group scheme over K, associated to (X,x) (see Section 4.5). Let $\Gamma(G^{(n)}(X,x))$ be the image of [1] under $G^{(n)}(X,x)^{op}:\mathrm{N}(\Delta)\to\mathrm{CAlg}_K$. (Namely, $\Gamma(G^{(n)}(X,x))$ is the underlying algebra of commutative Hopf algebra object $G^{(n)}(X,x)^{op}$ in CAlg_K.) Let $\mathsf{MG}_E=\mathrm{Spec}\,C$. The identity $\mathsf{MG}_E\to\mathsf{MG}_E$ determines a component of the space (∞-groupoid) $\mathsf{MG}_E(C)$. The action on $G^{(n)}(X,x)$ (cf. Proposition 7.12 and Example 7.15) induces its image in $\mathrm{Aut}(G^{(n)}(X,x))(C)$. The equivalence class of the image in $\mathrm{Aut}(G^{(n)}(X,x))(C)$ gives rise to a morphism $\Gamma(G^{(n)}(X,x))\otimes_K C\stackrel{\sim}{\to}\Gamma(G^{(n)}(X,x))\otimes_K C$ in CAlg_C (cf. Remark 7.2). Composing with the unit $K\to C$, we have

$$\theta:\Gamma(G^{(n)}(X,x))=\Gamma(G^{(n)}(X,x))\otimes_K K\to\Gamma(G^{(n)}(X,x))\otimes_K C\stackrel{\sim}{\to}\Gamma(G^{(n)}(X,x))\otimes_K C.$$

The composite is a coaction of C on $\Gamma(G^{(n)}(X,x))$ at the level of the homotopy category $\mathrm{h}(\mathrm{CAlg}_K)$. Namely, if we think of C as a coalgebra in $\mathrm{h}(\mathrm{CAlg}_K)$ determined by the class of comultiplication $C\to C\otimes_K C$ and the unit $C\to K$, then $\Gamma(G^{(n)}(X,x))\to\Gamma(G^{(n)}(X,x))\otimes_K C$ is an (associative) coaction on $\Gamma(G^{(n)}(X,x))$ in the obvious sense. Also, it commutes with the structure of coalgebra on $\Gamma(G^{(n)}(X,x))$ at the level of homotopy category. Let $B:=\tau C$ (see Section 7.3 for τ). Applying τ to θ we obtain

$$\rho:\tau(\Gamma(G^{(n)}(X,x)))\to\tau(\Gamma(G^{(n)}(X,x)))\otimes_K B\stackrel{\sim}{\to}\tau(\Gamma(G^{(n)}(X,x)))\otimes_K B.$$

Taking the cohomology in the 0-th term we have

$$\begin{aligned}\xi:H^0(\tau(\Gamma(G^{(n)}(X,x))))\to H^0(\tau(\Gamma(G^{(n)}(X,x)))\otimes_K B)&\stackrel{\sim}{\to}H^0(\tau(\Gamma(G^{(n)}(X,x)))\otimes_K B)\\&\simeq H^0(\tau(\Gamma(G^{(n)}(X,x)))\otimes H^0(B).\end{aligned}$$

Recall that the commutative Hopf ring $\Gamma(\overline{G}^{(n)}(X,x))$ of $\overline{G}^{(n)}(X,x)$ is $H^0(\tau(\Gamma(G^{(n)}(X,x))))$ equipped with the structure of commutative Hopf ring that comes from the structures on $\Gamma(G^{(n)}(X,x))$. Moreover, $MG_E = \operatorname{Spec} H^0(B)$. The morphism ξ is a coaction of $H^0(B)$ on the commutative K-algbera $H^0(\tau(\Gamma(G^{(n)}(X,x)))) = \Gamma(\overline{G}^{(n)}(X,x))$ which is compatible with the structure of coalgebra on the 0-th cohomology $H^0(\tau(\Gamma(G^{(n)}(X,x))))$. It gives rise to an action

$$\overline{G}^{(n)}(X,x) \times MG_E \to \overline{G}^{(n)}(X,x).$$

The functoriality (1) is obvious from the construction. □

Remark 7.19. The affine group scheme MG_E is a coarse moduli space for MG_E. It has a coarse moduli theoretic interpretation: for any field L over K, $\mathsf{MG}_E^{\circ} \to MG_E$ induces an isomorphism $\pi_0(\mathsf{MG}_E(L)) \xrightarrow{\sim} MG_E(L)$ of sets where $\pi_0(\mathsf{MG}_E(L))$ is the set of connected components, i.e., the set of equivalence classes of L-valued points on MG_E (cf. [24, Theorem 1.3]). By $\mathsf{MG}_E \simeq \operatorname{Aut}(\mathsf{R}_E)$, the set $MG_E(K)$ is naturally identified with the set of equivalence classes of the automorphism of $\mathsf{R}_E : \mathsf{DM}^{\otimes}(k) \to \operatorname{Mod}_K^{\otimes}$. Suppose that $q \in MG_E(K)$ corresponds to an automorphism σ of R_E. The automorphism of $\overline{G}^{(n)}(X,x)$ induced by q is the automorphism induced by σ. Recall that σ induces an automorphism of the multiplicative realization functor $\operatorname{CAlg}(\mathsf{R}_E) : \operatorname{CAlg}(\mathsf{DM}^{\otimes}(k)) \to \operatorname{CAlg}_K$ (cf. Section 7.1). It gives rise to an automorphism on $G^{(n)}(X,x)^{op} : \mathrm{N}(\Delta) \to \operatorname{CAlg}_K$ (cf. Section 7.2). The induced automorphism $\Gamma(G^{(n)}(X,x)) \xrightarrow{\sim} \Gamma(G^{(n)}(X,x))$ gives rise to $a : H^0(\tau\Gamma(G(X,x))) \xrightarrow{\sim} H^0(\tau\Gamma(G(X,x)))$. By our construction, the action of q is equal to a. This interpretation holds also for any field L over K.

8. Motivic homotopy exact sequence for algebraic curves

Let X be a geometrically connected scheme of finite type over a perfect field k and let $X_{\bar{k}}$ be the base change to a separable closure $\bar{k}$. Let G_k denote the absolute Galois group $\operatorname{Gal}(\bar{k}/k) = \pi_1^{\text{ét}}(\operatorname{Spec} k, \operatorname{Spec} \bar{k})$. We write $\pi_1^{\text{ét}}(-,a)$ for the étale fundamental group of "$(-)$" with a base point a. Let $\bar{x} : \operatorname{Spec} \bar{k} \to X_{\bar{k}}$ be a geometric point and let $x : \operatorname{Spec} \bar{k} \to X$ be the composite. There is an exact sequence of profinite groups

$$1 \to \pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x}) \to \pi_1^{\text{ét}}(X,x) \to G_k \to 1$$

induced by $X_{\bar{k}} = X \times_{\operatorname{Spec} k} \operatorname{Spec} \bar{k} \to X \to \operatorname{Spec} k$. It is usually called the homotopy exact sequence because it can be thought of as a fairly precise analogue of the long exact sequence that comes from a homotopy fiber sequence of topological spaces. The higher homotopy groups of étale homotopy type of $\operatorname{Spec} k$ in the sense of Artin-Mazur are trivial, and the above exact sequence may be understood as a part of a long exact sequence. In this section, combining the results of this paper with the Tannakian theory developed in [26] we formulate and prove a motivic counterpart of a homotopy exact sequence when X is a smooth curve (Proposition 8.12). The coefficient field of $\mathsf{DM}(k)$ and its full subcategories will be $\mathbb{Q}$, whereas K will be a coeffcient field of Weil cohomology theory.

Let C be a smooth curve, that is, a connected one dimensional smooth scheme separated of finite type over a perfect field k. Let $j : C \hookrightarrow \overline{C}$ be a smooth compactification of C. Namely, $\overline{C}$ is a smooth proper curve over k, and j is an open immersion with a dense image. Let Z denote the complement $\overline{C} - C$, that is a finite set of closed points $Z = p_0 \sqcup p_1 \sqcup \ldots \sqcup p_m$. For simplicity, we assume that $\overline{C}$ admits a k-rational point.

We begin by the definition of a symmetric monoidal full subcategory of $\mathsf{DM}^{\otimes}(k)$ that is "smaller" and more tractable than $\mathrm{CAlg}(\mathsf{DM}^{\otimes}(k))$.

Lemma 8.1. *Let A be an abelian variety over k and let l be a finite Galois extension of k. Let $\mathsf{DM}^{\otimes}(A, l/k)$ be the smallest symmetric monoidal stable full subcategory of $\mathsf{DM}^{\otimes}(k)$ which is closed under colimits and contains $M(A)$, the dual $M(A)^{\vee}$, $M(\mathrm{Spec}\, l)$ and Tate objects $\mathbf{1}_k(n)$ for any $n \in \mathbb{Z}$. (We remark that the symmetric monoidal structure on $\mathsf{DM}^{\otimes}(A, l/k)$ inherits from that of $\mathsf{DM}^{\otimes}(k)$, and $\mathsf{DM}(A, l/k)$ is presentable.)*

Let C be a smooth curve over k. Let k' be a Galois field extension of k such that for any $0 \le i \le m$, the residue field $k_i \supset k$ of p_i can be embedded into k'. Let $J_{\overline{C}}$ be the Jacobian variety of $\overline{C}$. Then M_C lies in $\mathrm{CAlg}(\mathsf{DM}^{\otimes}(J_{\overline{C}}, k'/k))$.

Proof. Since the underlying object $M_C \in \mathsf{DM}(k)$ is a dual of $M(C)$, it suffices to prove that $M(C)^{\vee}$ belongs to $\mathsf{DM}(J_{\overline{C}}, k'/k)$. We have a decomposition $M(J_{\overline{C}}) \simeq \oplus_{i=0}^{2g} M_i(J_{\overline{C}})$ for the Jacobian variety $J_{\overline{C}}$ such that $M_i(J_{\overline{C}}) \simeq \mathrm{Sym}^i(M_1(J_{\overline{C}}))$ (see Section 5.2). Here g is the genus of $\overline{C}$. Also, there is an isomorphism $M(\overline{C}) \simeq \mathbf{1}_k \oplus M_1(J_{\overline{C}}) \oplus \mathbf{1}_k(1)[2]$ in $\mathsf{DM}(k)$ (see e.g. [45], [42, 3.3.9]). Thus both $M(\overline{C})$ and $M(\overline{C})^{\vee} \simeq M(\overline{C}) \otimes \mathbf{1}_k(-1)[-2]$ lie in $\mathsf{DM}(J_{\overline{C}}, k'/k)$. By Gysin triangle (see [37, 14.5]), there is a distinguished triangle

$$M(C) \to M(\overline{C}) \to M(Z)(1)[2] \to$$

in the triangulated categories $\mathrm{h}(\mathsf{DM}(k))$. Therefore, we are reduced to showing that

$$M(Z)^{\vee} \simeq \oplus_{0 \le i \le m} M(\mathrm{Spec}\, k_i)^{\vee} \simeq \oplus_{0 \le i \le m} M(\mathrm{Spec}\, k_i)$$

belongs to $\mathsf{DM}(J_{\overline{C}}, k'/k)$. Using the functoriality with respect to finite correspondences, we deduce that each $M(\mathrm{Spec}\, k_i)$ is a direct summand of $M(\mathrm{Spec}\, k')$ (since we work with rational coefficients). □

The symmetric monoidal stable presentable ∞-category $\mathsf{DM}^{\otimes}(A, l/k)$ has a nice property: it is an *algebraic fine Tannakian ∞-category*. This notion has been introduced and studied in our work [26].

Proposition 8.2. *We follow the notation in Lemma 8.1. Let $M_1(A)$ be the direct summand of $M(A)$ in the decomposition in Section 5.2. Then $M = M_1(A)[-1] \oplus \mathbf{1}_k(1) \oplus M(\mathrm{Spec}\, l)$ is a wedge-finite object. Namely, there is an natural number n such that the wedge product $\wedge^{n+1} M$ is zero, and $\wedge^n M$ is an invertible object, see [26, Section 1]. Consequently, the symmetric monoidal ∞-category $\mathsf{DM}^{\otimes}(A, l/k)$ is an algebraic fine Tannakian ∞-category, see [26, Definition 4.4, Theorem 4.1].*

Proof. By [26, Proposition 6.1] and the fact that $\mathrm{Hom}_{\mathrm{h}(\mathsf{DM}(k))}(\mathbf{1}_k, \mathbf{1}_k) \simeq \mathbb{Q}$, it is enough to prove that the wedge product $\wedge^N M$ is zero for $N >> 0$. To this end, we are reduced to proving that $\wedge^N(M_1(A)[-1]) = 0$, $\wedge^N \mathbf{1}_k(1) = 0$, and $\wedge^N M(\mathrm{Spec}\, l) = 0$ for $N >> 0$. By the well-known Kimura finiteness (see [29], [2, Thereom7.1.1]), $\wedge^{2e+1}(M_1(A)[-1]) \simeq (\mathrm{Sym}^{2e+1} M_1(A))[-2e-1] \simeq 0$ where e is the dimension of A. Also, $\wedge^2 \mathbf{1}_k(1) = 0$ and $\wedge^{d+1} M(\mathrm{Spec}\, l) = 0$. Here $d = [l : k]$. The final claim follows from the definition of $\mathsf{DM}^{\otimes}(A, l/k)$ and the definition of algebraic fine Tannakian ∞-category. □

We define a derived stack from $\mathsf{DM}^{\otimes}(A, l/k)$ and $M = M_1(A)[-1] \oplus \mathbf{1}_k(1) \oplus M(\mathrm{Spec}\, l)$. By a derived stack over a field K, we mean a sheaf $\mathrm{CAlg}_K \to \widehat{\mathcal{S}}$ which satisfies a certain geometric condition. The ∞-category AlgSt_K of derived stacks is defined to be the full subcategory of $\mathrm{Fun}(\mathrm{CAlg}_K, \widehat{\mathcal{S}})$ that consists of derived stacks. A typical example is a derived affine scheme $\mathrm{Spec}\, R : \mathrm{CAlg}_K \to \widehat{\mathcal{S}}$, that is corepresented by $R \in \mathrm{CAlg}_K$. Thus there is a natural fully faithful embedding $\mathrm{Aff}_K \subset \mathrm{AlgSt}_K$. Another main example for us is a quotient stack $[\mathrm{Spec}\, R/G]$ that arises from an action of an algebraic affine group scheme G on $\mathrm{Spec}\, R$. We refer to [26, Section 2.1] for conventions and terminology concerning derived stacks.

Applying [26, Theorem 4.1] to $\mathsf{DM}^{\otimes}(A, l/k)$ with the wedge-finite object M we obtain

Corollary 8.3. *Let n be the natural number such that $\wedge^{n+1}M \simeq 0$ and $\wedge^n M$ is an invertible object. (Actually, one can see that $n = 2e + d + 1$ if e is the dimension of A, and $d = [l : k]$.) There exist a derived stack $\mathcal{X}_{A,l}$ over $\mathbb{Q}$ such that $\mathcal{X}_{A,l}$ has a presentation as a quotient stack of the form $[\mathrm{Spec}\, V_{A,l}/\mathrm{GL}_n]$ where $V_{A,l}$ is in $\mathrm{CAlg}_{\mathbb{Q}}$, and a symmetric monoidal $\mathbb{Q}$-linear equivalence*

$$\phi : \mathrm{QC}^{\otimes}(\mathcal{X}_{A,l}) \simeq \mathsf{DM}^{\otimes}(A, l/k).$$

Here GL_n is the general linear group over $\mathbb{Q}$ that acts on $V_{A,l}$, and $\mathrm{QC}^{\otimes}(\mathcal{X}_{A,l})$ is the symmetric monoidal $\mathbb{Q}$-linear presentable ∞-category of quasi-coherent complexes on $\mathcal{X}_{A,l}$. We shall call $\mathcal{X}_{A,l}$ the motivic Galois stack associated to $\mathsf{DM}^{\otimes}(A, l/k)$ and M. For the definition of $\mathrm{QC}(-)$, we refer to either [26, Section 2.3] or Remark 8.5.

Corollary 8.4. *We continue to use the notation in Lemma 8.1. Then M_C can be naturally regarded as a commutative object in $\mathrm{CAlg}(\mathrm{QC}^{\otimes}(\mathcal{X}_{J_{\overline{C}},k'}))$.*

Proof. Combine Lemma 8.1 and Corollary 8.3. □

Remark 8.5. For a quotient stack $[\mathrm{Spec}\, V/G]$ such that G is an algebraic affine group scheme, the symmetric monoidal ∞-category $\mathrm{QC}^{\otimes}([\mathrm{Spec}\, V/G])$ can be described in the following way. The action of G on $\mathrm{Spec}\, V$ can be defined by a simplicial diagram of derived affine schemes which is informally given by $[i] \mapsto \mathrm{Spec}\, V \times G^{\times i}$. If we put $\mathrm{Spec}\, R^i = \mathrm{Spec}\, V \times G^{\times i}$, then $\mathrm{QC}^{\otimes}([\mathrm{Spec}\, V/G])$ is defined to be $\varprojlim_{[i]} \mathrm{Mod}^{\otimes}_{R^i}$. The limit of the cosimplicial diagram $\{\mathrm{Mod}^{\otimes}_{R^n}\}_{[n]\in\Delta}$ is taken in the ∞-category of symmetric monoidal ∞-categories.

Remark 8.6. The stack $\mathcal{X}_{A,l} \simeq [\mathrm{Spec}\, V_{A,l}/\mathrm{GL}_n]$ is defined as follows (see [26] for details): Let $\mathrm{Rep}^{\otimes}(\mathrm{GL}_n)$ be the symmetric monoidal stable ∞-category of representations of GL_n (cf. Section 5). There is a canonical equivalence

$$\mathrm{QC}^{\otimes}([\mathrm{Spec}\, \mathbb{Q}/\mathrm{GL}_n]) \simeq \mathrm{Rep}^{\otimes}(\mathrm{GL}_n).$$

Since $[\mathrm{Spec}\, V_{A,l}/\mathrm{GL}_n]$ is affine over $B\mathrm{GL}_n := [\mathrm{Spec}\, \mathbb{Q}/\mathrm{GL}_n]$, $\mathrm{Spec}\, V_{A,l}$ with action of GL_n can be identified with an object in $\mathrm{CAlg}(\mathrm{Rep}^{\otimes}(\mathrm{GL}_n))$. By Proposition 5.2 we have a symmetric monoidal colimit-preserving functor $p : \mathrm{Rep}^{\otimes}(\mathrm{GL}_n) \to \mathsf{DM}^{\otimes}(A, l/k)$ which carries the standard representation of GL_n placed in degree zero to M. By the relative adjoint functor theorem, this functor admits a lax symmetric monoidal right adjoint $q : \mathsf{DM}^{\otimes}(A, l/k) \to \mathrm{Rep}^{\otimes}(\mathrm{GL}_n)$. Thus, q carries a unit object $\mathbf{1}_{\mathsf{DM}^{\otimes}(A,l/k)}$ to a commutative algebra object $U_{A,l} := q(\mathbf{1}_{\mathsf{DM}^{\otimes}(A,l/k)}) \in \mathrm{CAlg}(\mathrm{Rep}^{\otimes}(\mathrm{GL}_n))$. This object $U_{A,l}$ amounts to $V_{A,l}$ endowed with action of GL_n, i.e., data of $[\mathrm{Spec}\, V_{A,l}/\mathrm{GL}_n]$. The commutative algebra $V_{A,l}$ in $\mathrm{CAlg}_{\mathbb{Q}}$ is the image of $U_{A,l}$ in $\mathrm{CAlg}_{\mathbb{Q}}$. We remark that there is a canonical equivalence $\mathrm{QC}^{\otimes}([\mathrm{Spec}\, V_{A,l}/\mathrm{GL}_n]) \simeq \mathrm{Mod}^{\otimes}_{U_{A,l}}(\mathrm{Rep}^{\otimes}(\mathrm{GL}_n))$

where $\mathrm{Mod}^{\otimes}_{U_{A,l}}(\mathrm{Rep}^{\otimes}(\mathrm{GL}_n))$ is the symmetric monoidal ∞-category of $U_{A,l}$-module objects in $\mathrm{Rep}^{\otimes}(\mathrm{GL}_n)$. This equivalence makes the diagram

$$\begin{array}{ccccc} \mathrm{QC}^{\otimes}(B\mathrm{GL}_n) & \longrightarrow & \mathrm{QC}^{\otimes}([\mathrm{Spec}\, V_{A,l}/\mathrm{GL}_n]) & & \\ \downarrow \simeq & & \downarrow \simeq & \searrow^{\phi} & \\ \mathrm{Rep}^{\otimes}(\mathrm{GL}_n) & \xrightarrow[\otimes U_{A,l}]{} & \mathrm{Mod}^{\otimes}_{U_{A,l}}(\mathrm{Rep}^{\otimes}(\mathrm{GL}_n)) & \longrightarrow & \mathsf{DM}^{\otimes}(A, l/k). \end{array}$$

commute up to homotopy, where the top horizontal arrow is the pullback functor of the projection $[\mathrm{Spec}\, V_{A,l}/\mathrm{GL}_n] \to B\mathrm{GL}_n$. The equivalence $\mathrm{Mod}^{\otimes}_{U_{A,l}}(\mathrm{Rep}^{\otimes}(\mathrm{GL}_n)) \to \mathsf{DM}^{\otimes}(A, l/k)$ is defined to be the composite

$$\begin{aligned} \mathrm{Mod}^{\otimes}_{U_{A,l}}(\mathrm{Rep}^{\otimes}(\mathrm{GL}_n)) &\to \mathrm{Mod}^{\otimes}_{p(U_{A,l})}(\mathsf{DM}^{\otimes}(A, l/k)) \\ &\to \mathrm{Mod}^{\otimes}_{\mathbf{1}_{\mathsf{DM}^{\otimes}(A,l/k)}}(\mathsf{DM}^{\otimes}(A, l/k)) \simeq \mathsf{DM}^{\otimes}(A, l/k) \end{aligned}$$

where the first functor is induced by p, and the second functor is induced by the base change along the counit map $p(U_{A,l}) = pq(\mathbf{1}_{\mathsf{DM}^{\otimes}(A,l/k)}) \to \mathbf{1}_{\mathsf{DM}^{\otimes}(A,l/k)}$. The composite of lower horizontal arrows is equivalent to p.

Remark 8.7. There is the following uniqueness. Let $(\mathcal{Y}, N)$ be a pair that consists of a derived stack $\mathcal{Y}$ over $\mathbb{Q}$, and N is a vector bundle on $\mathcal{Y}$. Here by a vector bundle we mean an object N in $\mathrm{QC}(\mathcal{Y})$ such that for any $f : \mathrm{Spec}\, R \to \mathcal{Y}$, the restriction $f^*(N)$ is equivalent to a direct summand of some finite coproduct $R^{\oplus m}$. The stack $\mathcal{X}_{A,l} \simeq [\mathrm{Spec}\, V_{A,l}/\mathrm{GL}_n]$ has a vector bundle $N_{A,l}$ that is defined to be the pullback of the tautological vector bundle on $BGL_n = [\mathrm{Spec}\, \mathbb{Q}/\mathrm{GL}_n]$. So we have such a pair $(\mathcal{X}_{A,l}, N_{A,l})$. By the diagram in Remark 8.6, the equivalence $\phi : \mathrm{QC}^{\otimes}(\mathcal{X}_{A,l/k}) \simeq \mathsf{DM}^{\otimes}(A, l/k)$ sends $N_{A,l}$ to M. Assume that there is a symmetric monoidal $\mathbb{Q}$-linear equivalence $\mathrm{QC}^{\otimes}(\mathcal{Y}) \simeq \mathsf{DM}^{\otimes}(A, l/k)$ which sends N to M. Then there is an equivalence $\mathcal{Y} \simeq \mathcal{X}_{A,l}$ such that the induced equivalence $\mathrm{QC}^{\otimes}(\mathcal{Y}) \simeq \mathrm{QC}^{\otimes}(\mathcal{X}_{A,l})$ sends N to $N_{A,l}$. This uniqueness will not be necessary in this paper, so that we will not present the proof. But one can prove it by using arguments in [26].

We say that a morphism $\mathcal{X} \to \mathcal{Y}$ of derived stacks over K is affine if, for any $\mathrm{Spec}\, R \to \mathcal{Y}$ from a derived affine scheme, the fiber product $\mathcal{X} \times_{\mathcal{Y}} \mathrm{Spec}\, R$ belongs to Aff_K. Let $\mathrm{Aff}_{\mathcal{Y}}$ be the full subcategory of the overcategory $(\mathrm{AlgSt}_K)_{/\mathcal{Y}}$ that consists of affine morphisms $\mathcal{X} \to \mathcal{Y}$. There is a canonical equivalence $\mathrm{Aff}_{\mathcal{Y}} \simeq \mathrm{CAlg}(\mathrm{QC}^{\otimes}(\mathcal{Y}))^{op}$ (cf. [26, Section 2.3], this is a direct generalization of the analogous fact in the usual scheme theory).

Definition 8.8. By Corollary 8.4, let us consider M_C as an object in $\mathrm{CAlg}(\mathrm{QC}^{\otimes}(\mathcal{X}_{J_{\overline{C}},k'}))$. Let $\mathcal{M}_C \to \mathcal{X}_{J_{\overline{C}},k'}$ be a derived stack affine over $\mathcal{X}_{J_{\overline{C}},k'}$ that corresponds to M_C through the equivalence $\mathrm{Aff}_{\mathcal{X}_{J_{\overline{C}},k'}} \simeq \mathrm{CAlg}(\mathrm{QC}^{\otimes}(\mathcal{X}_{J_{\overline{C}},k'}))^{op}$.

Let $\mathsf{R}_E : \mathsf{DM}^{\otimes}(k) \to \mathsf{D}^{\otimes}(K) \simeq \mathrm{Mod}^{\otimes}_K$ be the realization functor associated to a mixed Weil Theory E with coefficients in a field K of characteristic zero. By abuse of notation we write R_E also for the restriction $\mathsf{DM}^{\otimes}(A, l/k) \to \mathsf{D}^{\otimes}(K)$. Suppose that $\mathsf{R}_E(M)$ is concentrated in degree zero $\mathsf{D}(K)$ (all known mixed Weil theories satisfy this condition). As discussed in [27, Section 4.1] or [26, Remark 6.12], it gives rise to a morphism

$$\rho_E : \mathrm{Spec}\, K \to \mathcal{X}_{A,l}.$$

We refer to this morphism as the base point of R_E.

We briefly recall the construction of ρ_E. Let $p : \mathrm{QC}^{\otimes}(B\mathrm{GL}_n) \simeq \mathrm{Rep}^{\otimes}(\mathrm{GL}_n) \to \mathsf{DM}^{\otimes}(A, l/k)$ be the sequence contained in the diagram in Remark 8.6. Note that this functor carries the standard representation of GL_n placed in degree zero to M, and the realization functor carries M to the n-dimensional vector space placed in degree zero in $\mathsf{D}(K)$. Therefore, by the universal property of $\mathrm{Rep}^{\otimes}(\mathrm{GL}_n)$ (Proposition 5.2 or [26, Theorem 3.1]), the composite $\mathrm{QC}^{\otimes}(B\mathrm{GL}_n) \simeq \mathrm{Rep}^{\otimes}(\mathrm{GL}_n) \to \mathsf{DM}^{\otimes}(A, l/k) \to \mathsf{D}^{\otimes}(K)$ is equivalent to the pullback functor $\mathrm{QC}^{\otimes}(B\mathrm{GL}_n) \to \mathsf{D}^{\otimes}(K) \simeq \mathrm{QC}^{\otimes}(\mathrm{Spec}\, K)$ along $\mathrm{Spec}\, K \to \mathrm{Spec}\, \mathbb{Q} \to B\mathrm{GL}_n$. Let $u : \mathsf{D}^{\otimes}(K) \to \mathrm{Rep}^{\otimes}(\mathrm{GL}_n) \simeq \mathrm{QC}^{\otimes}(B\mathrm{GL}_n)$ be the lax symmetric monoidal right adjoint to $\mathrm{QC}^{\otimes}(B\mathrm{GL}_n) \to \mathrm{QC}^{\otimes}(\mathrm{Spec}\, K) \simeq \mathsf{D}^{\otimes}(K)$, whose existence is ensured by the relative adjoint functor theorem. Then this right adjoint induces $\mathrm{CAlg}(\mathsf{D}^{\otimes}(K)) \simeq \mathrm{CAlg}_K \to \mathrm{CAlg}(\mathrm{Rep}^{\otimes}(\mathrm{GL}_n))$ which carries the unit algebra K to $u(K) \simeq \Gamma(\mathrm{GL}_n) \otimes_{\mathbb{Q}} K \in \mathrm{CAlg}(\mathrm{Rep}^{\otimes}(\mathrm{GL}_n)) \simeq \mathrm{CAlg}(\mathrm{QC}^{\otimes}(B\mathrm{GL}_n))$. Here, write $\Gamma(\mathrm{GL}_n)$ for the (ordinary) coordinate ring of the general linear group GL_n which is endowed with the natural action of GL_n. The symbol K in $\Gamma(\mathrm{GL}_n) \otimes_{\mathbb{Q}} K$ is understood as the $\mathbb{Q}$-algebra K with the trivial action of GL_n. Note that there is a natural morphism $U_{A,l} \to u(K) \simeq \Gamma(\mathrm{GL}_n) \otimes_{\mathbb{Q}} K$ in $\mathrm{CAlg}(\mathrm{QC}^{\otimes}(B\mathrm{GL}_n))$. In fact, if $v : \mathrm{CAlg}_K \to \mathrm{CAlg}(\mathsf{DM}^{\otimes}(A, l/k))$ denotes the right adjoint to the restricted multiplicative realization functor $\mathrm{CAlg}(\mathsf{DM}^{\otimes}(A, l/k)) \to \mathrm{CAlg}_K$, then there is a unit map $\mathbf{1}_{\mathsf{DM}(A,l/k)} \to v(K)$ that induces $U_{A,l} = q(\mathbf{1}_{\mathsf{DM}(A,l/k)}) \to qv(K) = u(K)$, as claimed (for the functor q, see Remark 8.6). By using the equivalence $\mathrm{Aff}_{B\mathrm{GL}_n} \simeq \mathrm{CAlg}(\mathrm{QC}^{\otimes}(B\mathrm{GL}_n))^{op}$, we obtain $\rho_E : \mathrm{Spec}\, K \simeq [\mathrm{Spec}\, \Gamma(\mathrm{GL}_n) \otimes_{\mathbb{Q}} K/\mathrm{GL}_n] \to \mathcal{X}_{A,l} = [\mathrm{Spec}\, V_{A,l}/\mathrm{GL}_n]$.

Remark 8.9. By this construction and Remark 8.6, we see that the diagram

$$\begin{array}{ccc} \mathrm{QC}^{\otimes}(\mathcal{X}_{A,l}) & \xrightarrow{\rho_E^*} & \mathrm{QC}^{\otimes}(\mathrm{Spec}\, K) \\ \simeq \downarrow \phi & & \downarrow \simeq \\ \mathsf{DM}^{\otimes}(A, l/k) & \xrightarrow{\mathsf{R}_E} & \mathsf{D}^{\otimes}(K) \end{array}$$

commutes up to homotopy, where ρ_E^* is the pullback functor (cf. [26, Section 2.3]), the right vertical arrow is a canonical equivalence.

One can associate to the base point $\rho_E : \mathrm{Spec}\, K \to \mathcal{X}_{A,l}$ a derived affine group scheme over K. Namely, we take the Cech nerve $G : \mathrm{N}(\Delta_+)^{op} \to \mathrm{AlgSt}_K$ of $\rho_E \times \mathrm{id} : \mathrm{Spec}\, K \to \mathcal{X}_{A,l} \times_{\mathrm{Spec}\, \mathbb{Q}} \mathrm{Spec}\, K$, which is defined to be the right Kan extension $\mathrm{N}(\Delta_+)^{op} \to \mathrm{AlgSt}_K$ of $\mathrm{N}(\Delta_+^{\le 0})^{op} = \mathrm{N}(\{[-1] \to [0]\})^{op} \to \mathrm{AlgSt}_K$ determined by $\rho_E \times \mathrm{id}$. The evaluation $G([1])$ is equivalent to $\mathrm{Spec}\, K \times_{\mathcal{X}_{A,l} \times \mathrm{Spec}\, K} \mathrm{Spec}\, K$ which is affine because the diagonal $[\mathrm{Spec}\, V_{A,l}/\mathrm{GL}_n] \to [\mathrm{Spec}\, V_{A,l}/\mathrm{GL}_n] \times [\mathrm{Spec}\, V_{A,l}/\mathrm{GL}_n]$ is affine. Thus the restriction of G defines a group object $\mathrm{N}(\Delta)^{op} \to \mathrm{Aff}_K$, whose underlying derived affine scheme is $\mathrm{Spec}\, K \times_{\mathcal{X}_{A,l} \times \mathrm{Spec}\, K} \mathrm{Spec}\, K$. We write $\Omega_{\rho_E} \mathcal{X}_{A,l}$ for this derived affine group scheme over K. The derived group scheme $\Omega_{\rho_E} \mathcal{X}_{A,l}$ is related to the derived motivic Galois group:

Proposition 8.10. *Let $\mathsf{MG}_{E, \mathsf{DM}^{\otimes}(A,l/k)}$ be the derived motivic Galois group which represents the automorphism group functor $\mathrm{Aut}(\mathsf{R}_E|_{\mathsf{DM}^{\otimes}(A,l/k)}) : \mathrm{CAlg}_K \to \mathrm{Grp}(\widehat{\mathcal{S}})$, cf. Remark 7.8. Then $\Omega_{\rho_E} \mathcal{X}_{A,l}$ is naturally equivalent to $\mathsf{MG}_{E, \mathsf{DM}^{\otimes}(A,l/k)}$.*

Proof. By Remark 8.9, we have $\mathrm{Aut}(\mathsf{R}_E|_{\mathsf{DM}^{\otimes}(A,l/k)}) \simeq \mathrm{Aut}(\rho_E^*)$ where $\rho_E^* : \mathrm{QC}^{\otimes}(\mathcal{X}_{A,l}) \to \mathrm{QC}^{\otimes}(\mathrm{Spec}\, K)$. It will suffice to show that $\Omega_{\rho_E} \mathcal{X}_{A,l} \simeq \mathrm{Aut}(\rho_E^*)$. This equivalence follows from [25, Proposition 4.6]. □

Remark 8.11. By the representability of automorphism groups, the restriction to $\mathsf{DM}^{\otimes}(A, l/k)$ induces $\mathsf{MG}_E \to \mathsf{MG}_{E,\mathsf{DM}^{\otimes}(A,l/k)} \simeq \Omega_{\rho_E}\mathcal{X}_{A,l}$. The action of MG_E on $\mathsf{R}_E(M_C)$ described in Proposition 7.5 factors through $\mathsf{MG}_E \to \Omega_{\rho_E}\mathcal{X}_{A,l}$.

Now we are ready to prove the following pullback diagram which can be regarded as a motivic generalization of the homotopy exact sequence for the étale fundamental group of C

$$1 \to \pi_1^{\text{ét}}(C_{\bar{k}}, \bar{c}) \to \pi_1^{\text{ét}}(C, c) \to G_k \to 1$$

(see Remark 8.13).

Proposition 8.12. *Let $\mathcal{M}_C \to \mathcal{X}_{J_{\overline{C}},k'/k}$ be the affine morphism defined in Definition 8.8. Let us consider the pullback diagram of derived stacks*

$$\begin{array}{ccc} \mathcal{F}_E & \longrightarrow & \mathcal{M}_C \\ \downarrow & & \downarrow \\ \operatorname{Spec} K & \xrightarrow{\rho_E} & \mathcal{X}_{J_{\overline{C}},k'} \end{array}$$

in $\mathrm{AlgSt}_{\mathbb{Q}}$. *(One may think of this diagram as a Cartesian diagram in* $\mathrm{Fun}(\mathrm{CAlg}_{\mathbb{Q}}, \widehat{\mathcal{S}})$.*) Then the fiber $\mathcal{F}_E$ is naturally equivalent to* $\operatorname{Spec} \mathsf{R}_E(M_C)$, *where* $\mathsf{R}_E(M_C)$ *in* CAlg_K *is the image of M_C under the multiplicative realization functor* $\mathsf{R}_E : \mathrm{CAlg}(\mathsf{DM}^{\otimes}(k)) \to \mathrm{CAlg}_K$. *In particular, when E is the singular cohomology theory, by Theorem 4.3 we have a Cartesian diagram*

$$\begin{array}{ccc} \operatorname{Spec} A_{PL,\infty}(C^t) & \longrightarrow & \mathcal{M}_C \\ \downarrow & & \downarrow \\ \operatorname{Spec} \mathbb{Q} & \xrightarrow{\rho_E} & \mathcal{X}_{J_{\overline{C}},k'}. \end{array}$$

Remark 8.13. The morphism $\mathcal{M}_C \to \mathcal{X}_{J_{\overline{C}},k'}$ should be thought of as a motivic counterpart of the delooping of $\pi_1^{\text{ét}}(C, c) \to G_k$. By Proposition 8.10 we can obtain the derived motivic Galois group $\mathsf{MG}_{E,\mathsf{DM}^{\otimes}(J_{\overline{C}},k'/k)} \simeq \Omega_{\rho_E}\mathcal{X}_{J_{\overline{C}},k'}$ from the base stack $\operatorname{Spec} K \to \mathcal{X}_{J_{\overline{C}},k'}$ by using the construction of the base loop space. The fiber $\mathcal{F}_E$ shoud be understood as a role of the delooping of $\pi_1^{\text{ét}}(C_{\bar{k}}, \bar{c})$. Consider the situation that k is a subfield of $\mathbb{C}$. Then $\pi_1^{\text{ét}}(C_{\mathbb{C}}, \bar{c})$ is isomorphic to the profinite completion of the topological fundamental group $\pi_1(C^t, \bar{c})$ of the underlying topological space C^t of $C_{\mathbb{C}} = C \times_{\operatorname{Spec} k} \operatorname{Spec} \mathbb{C}$. On the other hand, if we fix a k-rational point c, the unipotent group scheme $\overline{G}^{(1)}(C, c) \simeq \operatorname{Spec} H^0(\mathbb{Q} \otimes_{A_{PL,\infty}(C^t)} \mathbb{Q})$ is the pro-unipotent completion of the topological fundamental group $\pi_1(C^t, c)$.

Proof. We have already done all things. By Remark 8.9, one can identify the multiplicative realization functor $\mathrm{CAlg}(\mathsf{DM}^{\otimes}(J_{\overline{C}}, k'/k)) \to \mathrm{CAlg}_K$ with

$$\mathrm{CAlg}(\mathrm{QC}^{\otimes}(\mathcal{X}_{J_{\overline{C}},k'})) \to \mathrm{CAlg}(\mathrm{QC}^{\otimes}(\operatorname{Spec} K))$$

induced by the pullback functor ρ_E^*. Then we use the observation that the canonical equivalences

$$\mathrm{CAlg}(\mathrm{QC}^{\otimes}(\mathcal{X}_{J_{\overline{C}},k'}))^{op} \simeq \mathrm{Aff}_{\mathcal{X}_{J_{\overline{C}},k'}} \quad \text{and} \quad \mathrm{CAlg}(\mathrm{QC}^{\otimes}(\operatorname{Spec} K))^{op} \simeq \mathrm{Aff}_{\operatorname{Spec} K}$$

are compatible with pullback functors. Namely, through these canonical equivalences, the opposite functor $\mathrm{CAlg}(\mathrm{QC}^{\otimes}(\mathcal{X}_{J_{\overline{C}},k'}))^{op} \to \mathrm{CAlg}(\mathrm{QC}^{\otimes}(\operatorname{Spec} K))^{op}$ can be identified with $\mathrm{Aff}_{\mathcal{X}_{J_{\overline{C}},k'}} \to \mathrm{Aff}_{\operatorname{Spec} K} = \mathrm{Aff}_K$ given by $\{\mathcal{Z} \to \mathcal{X}_{J_{\overline{C}},k'}\} \mapsto \{\mathrm{pr}_2 : \mathcal{Z} \times_{\mathcal{X}_{J_{\overline{C}},k'}} \operatorname{Spec} K \to \operatorname{Spec} K\}$. Therefore, we see that $\mathcal{F}_E$ is equivalent to $\operatorname{Spec} \mathsf{R}_E(M_C)$ via these identifications. □

Appendix A: Comparison results

We will compare the motivic algebra of path torsors with an approach by Deligne-Goncharov [14].

A.1 Suppose that k is a number field. We work with rational coefficients. We begin by reviewing the category of mixed Tate motives over k. Let $\mathsf{DTM} := \mathsf{DTM}(k)$ be the smallest stable subcategory of $\mathsf{DM}(k)$ that is closed under small colimits and consists of $\mathbf{1}_k(n)$ for any $n \in \mathbb{Z}$. The stable subcategory DTM inherits a symmetric monoidal structure from $\mathsf{DM}(k)$. We refer to it as the symmetric monoidal stable ∞-category of mixed Tate motives and denote it by $\mathsf{DTM}^{\otimes}$. The stable ∞-category DTM is compactly generated. Let $\mathsf{DTM}_{\vee}$ denote the stable subcategory spanned by compact objects. In particular, $\mathrm{Ind}(\mathsf{DTM}_{\vee}) \simeq \mathsf{DTM}$ where $\mathrm{Ind}(-)$ indicates the Ind-category. The full subcategory $\mathsf{DTM}_{\vee}$ coincides with the stable subcategory consisting of dualizable objects. Let $(\mathsf{D}(\mathbb{Q})_{\geq 0}, \mathsf{D}(\mathbb{Q})_{\leq 0})$ be the standard t-structure on $\mathsf{D}(\mathbb{Q})$ such that C belongs to $\mathsf{D}(\mathbb{Q})_{\geq 0}$ (resp. $\mathsf{D}(\mathbb{Q})_{\leq 0}$) if and only if $H^{-i}(C) = H_i(C) = 0$ for $i < 0$ (resp. $i > 0$). For our conventions on (motivic) t-structures, we refer to [33] and [25, Section 7]. Under the setting where k is a number field, there is a nondegenerate bounded t-structure on $\mathsf{DTM}_{\vee}$ given by

$$\mathsf{DTM}_{\vee,\geq 0} := \mathsf{R}_T^{-1}(\mathsf{D}(\mathbb{Q})_{\geq 0}) \cap \mathsf{DTM}_{\vee}, \quad \mathsf{DTM}_{\vee,\leq 0} := \mathsf{R}_T^{-1}(\mathsf{D}(\mathbb{Q})_{\leq 0}) \cap \mathsf{DTM}_{\vee}$$

where $\mathsf{R}_T : \mathsf{DTM}^{\otimes} \to \mathsf{D}^{\otimes}(\mathbb{Q})$ is the singular realization functor. It is the motivic t-structure on $\mathsf{DTM}_{\vee}$ (cf. [30]). The realization functor $\mathsf{DTM}_{\vee} \to \mathsf{D}(\mathbb{Q})$ is t-exact and conservative. The both categories $\mathsf{DTM}_{\vee,\geq 0}$ and $\mathsf{DTM}_{\vee,\leq 0}$ are closed under tensor products. Let $\mathsf{TM}^{\otimes}$ be the heart $\mathsf{DTM}_{\vee,\geq 0} \cap \mathsf{DTM}_{\vee,\leq 0}$ which is a symmetric monoidal (furthermore Tannakian) abelian category. We refer to $\mathsf{TM}^{\otimes}$ as the abelian category of mixed Tate motives.

A.2 The construction in Deligne-Goncharov [14] employs the idea in Wojtkowiak [51] that uses cosimplicial schemes. Let X be a smooth variety over k. Let $x : \operatorname{Spec} k \to X$ and $y : \operatorname{Spec} k \to X$ be two k-rational points. To (X, x, y) we associate a cosimplicial smooth scheme, i.e., a functor $P^{\Delta}(X, x, y) : \Delta \to \mathrm{Sm}_k : [n] \mapsto X^n$ whose cofaces are defined by

$$d^0(x_1, \ldots, x_n) = (x_1, \ldots, x_n, x), \quad d^{n+1}(x_1, \ldots, x_n) = (y, x_1, \ldots, x_n),$$
$$d^i(x_1, \ldots, x_n) = (x_1, \ldots, x_{n-i+1}, x_{n-i+1}, \ldots, x_n), \quad (0 < i < n),$$

$d^0, d^1 : X^0 = \operatorname{Spec} k \rightrightarrows X^1 = X$ is given by x and y. The codegeneracy are given by projections. Recall the functor $\Xi : \mathrm{Sm}_k^{op} \to \mathrm{CAlg}(\mathsf{DM}^{\otimes}(k))$ from Section 3.2. By abuse of notation we write Ξ for the composite $\mathrm{Sm}_k^{op} \stackrel{\Xi}{\to} \mathrm{CAlg}(\mathsf{DM}^{\otimes}(k)) \to \mathsf{DM}(k)$ where the second functor is the forgetful functor. Consider the simplicial object in $\mathsf{DM}(k)$ given by the composition

$$\mathcal{M}_{\Delta}(X, x, y) : \mathrm{N}(\Delta)^{op} \stackrel{P^{\Delta}(X,x,y)^{op}}{\longrightarrow} \mathrm{Sm}_k^{op} \stackrel{\Xi}{\to} \mathsf{DM}(k).$$

Let Δ_s be the subcategory of Δ whose objects coincide with that of Δ, and whose morphisms are injective maps. The inclusion $\mathrm{N}(\Delta_s)^{op} \hookrightarrow \mathrm{N}(\Delta)^{op}$ is cofinal [32, 6.5.3.7]. It follows that a colimit of $\mathcal{M}_{\Delta}(X, x, y)$ is naturally equivalent to that of the restriction

$$\mathcal{M}_{\Delta}(X, x, y)|_{\mathrm{N}(\Delta_s)^{op}} : \mathrm{N}(\Delta_s)^{op} \to \mathsf{DM}(k).$$

Let $\Delta_{s,\le n}$ be the full subcategory of Δ_s spanned by $\{[0],\dots,[n]\}$. Let $\mathcal{M}_{\Delta_{s,\le n}}(X,x,y):\mathrm{N}(\Delta_{s,\le n})^{op}\to\mathsf{DM}(k)$ denote the restriction of $\mathcal{M}_\Delta(X,x,y)$. Let $\mathcal{M}(X,x,y)$ denote a colimit of $\mathcal{M}_\Delta(X,x,y)|_{\mathrm{N}(\Delta_s)^{op}}$ (or equivalently $\mathcal{M}_\Delta(X,x,y)$). Let $\mathcal{M}_n(X,x,y)$ denote a colimit of $\mathcal{M}_{\Delta_{s,\le n}}(X,x,y)$ in $\mathsf{DM}(k)$. The colimits $\mathcal{M}_n(X,x,y)$ naturally constitute a sequence $\mathcal{M}_0(X,x,y)\to\mathcal{M}_1(X,x,y)\to\cdots$, and there is a canonical equivalence $\varinjlim_n\mathcal{M}_n(X,x,y)\simeq\mathcal{M}(X,x,y)$ (cf. [32, 4.2.3]). Now suppose that $M(X)$ belongs to $\mathsf{DTM}_\vee$. Then $M_{X^r}\simeq(M(X)^{\otimes r})^\vee$ lies in $\mathsf{DTM}_\vee$. Consequently, the finite colimit $\mathcal{M}_n(X,x,y)$ belongs to $\mathsf{DTM}_\vee$. Take the 0-th cohomology $H^0(\mathcal{M}_n(X,x,y))$ with respect to motivic t-structure. We let

$$M_{DG}(X,x,y):=\varinjlim_n H^0(\mathcal{M}_n(X,x,y))$$

be the filtered colimit in $\mathrm{Ind}(\mathsf{TM})$. We refer to it as the Deligne-Goncharov motive associated to (X,x,y). By [25, 7.4], $\mathsf{DTM}\simeq\mathrm{Ind}(\mathsf{DTM}_\vee)$ has a t-structure given by

$$(\mathrm{Ind}(\mathsf{DTM}_{\vee,\ge 0}),\mathrm{Ind}(\mathsf{DTM}_{\vee,\le 0})).$$

Passing to the 0-th cohomology (with respect to t-structure) commutes with filtered colimits so that $M_{DG}(X,x,y)=\varinjlim_n H^0(\mathcal{M}_n(X,x,y))\simeq H^0(\mathcal{M}(X,x,y))$. Therefore $M_{DG}(X,x,y)$ is nothing else but the 0-th cohomology of a colimit of the simplicial diagram $\mathcal{M}_\Delta(X,x,y)$.

Remark A.1. Taking advantage of a functorial assignment $X\mapsto M_X$ (see Proposition 3.4), we here give the cohomological construction of $M_{DG}(X,x,y)$ while the homological one is described in [14, 3.12]. Thus, procedures are dual to one another. In *loc. cit.*, one considers the diagram $\mathrm{N}(\Delta_{s,\le n})\to\mathsf{DM}(k):[r]\mapsto M(X^r)$ induced by the restricted diagram $P^{\Delta_{s,\le n}}(X,x,y):\mathrm{N}(\Delta_{s,\le n})\to\mathrm{Sm}_k:[r]\mapsto X^r$ instead of $\mathcal{M}_{\Delta_{s,\le n}}(X,x,y)$ (see [14, 3.12]). Then take a finite limit of the diagram in $\mathsf{DM}(k)$ by means of Moore complexes. The pleasant feature of cohomological construction is that it is not necessary to take the family of the restricted diagrams (though we take trouble to take them): one can directly define it to be the 0-th cohomology of a colimit of the simplicial diagram $\mathcal{M}_\Delta(X,x,y)$.

Remark A.2. One can consider a larger subcategory that consists of Artin-Tate motives. This category contains not only Tate motives but also motives of the form $M(\operatorname{Spec}k')$ such that k' is a finite separable extension field of k. We can treat this category by using a main result of [18] and [25, Section 8]. But we will not pursue a generalizaton to this direction.

A.3 We will think of $\mathsf{TM}^\otimes$ as a neutral tannakian category over $\mathbb{Q}$, which is endowed with the (symmetric monoidal) singular realization functor to the category of vector spaces over $\mathbb{Q}$

$$R_T:\mathsf{TM}^\otimes\to\mathrm{Vect}_{\mathbb{Q}}^\otimes.$$

The Tannaka dual MTG with respect to this functor is a pro-algebraic group over $\mathbb{Q}$ which represents the automorphism group of this symmetric monoidal functor R_T. For any $M\in\mathsf{TM}$ $MTG\simeq\mathrm{Aut}(R_T)$ naturally acts on $R_T(M)$. It gives rise to a $\mathbb{Q}$-linear symmetric monoidal equivalence $\mathsf{TM}^\otimes\simeq\mathrm{Rep}^\otimes(MTG)_\vee$ where $\mathrm{Rep}^\otimes(MTG)_\vee$ is the symmetric monoidal abelian category of finite dimensional representations of MTG. Recall from [25] the relation of tannakization and MTG.

Proposition A.3 (cf. Theorem 7.16 in [25])**.** *Let* MTG *be the derived affine group scheme which represents the automorphism group of* $\mathsf{R}_T:\mathsf{DTM}^\otimes\to\mathsf{D}^\otimes(\mathbb{Q})$*, that is, the tannakziation of* $\mathsf{R}_T:\mathsf{DTM}_\vee^\otimes\to\mathsf{D}^\otimes(\mathbb{Q})$ *in the sense of [24]. Then there is a natural isomorphism between MTG and the underlying group scheme of* MTG.

Remark A.4. There are approaches to MTG by means of bar constructions, see Spitzweck's derived tannakian presentation of $\mathsf{DTM}^{\otimes}$ [47], (see also [25], [26]). If we suppose futhermore that k is a number field, then by Borel's computation of rational motivic cohomology groups of number fields, it is not difficult to prove that $\mathsf{MTG} \simeq MTG$.

Let X be a smooth variety and assume that $M(X)$ belongs to $\mathsf{DTM}_{\vee}$ (thus M_X also lies in $\mathsf{DTM}_{\vee}$). Let $x, y : \mathrm{Spec}\, k \rightrightarrows X$ be two k-rational points on X. Recall the motivic algebra of path torsors

$$P_X(x,y) = \mathbf{1}_k \otimes_{M_X} \mathbf{1}_k$$

in $\mathrm{CAlg}(\mathsf{DTM}^{\otimes}) \subset \mathrm{CAlg}(\mathsf{DM}^{\otimes}(k))$ from Example 3.12. Take the cohomology $H^0(\mathbf{1}_k \otimes_{M_X} \mathbf{1}_k)$ with respect to the t-structure $(\mathrm{Ind}(\mathsf{DTM}_{\vee,\geq 0}), \mathrm{Ind}(\mathsf{DTM}_{\vee,\leq 0}))$.

Proposition A.5. *The cohomology $H^0(P_X(x,y))$ inherits the structure of commutative algebra object in* $\mathrm{Ind}(\mathsf{TM})$ *from $P_X(x,y)$. (The construction is described in the proof below.)*

Proof. Note first that M_X is the dual of $M(X)$ in DTM, and $M(X)$ belongs to $\mathsf{DTM}_{\vee,\geq 0}$. Since $\mathsf{R}_T(M_X)$ is the dual of $\mathsf{R}_T(M(X)) \in \mathsf{D}(\mathbb{Q})_{\geq 0}$, thus M_X lies in $\mathsf{DTM}_{\vee,\leq 0}$. Remember that $\mathsf{R}_T : \mathrm{CAlg}(\mathsf{DTM}^{\otimes}) \to \mathrm{CAlg}(\mathsf{D}^{\otimes}(\mathbb{Q}))$ is a left adjoint (in particular, it preserves colimits). It follows that $\mathsf{R}_T(\mathbf{1}_k \otimes_{M_X} \mathbf{1}_k) \simeq \mathbb{Q} \otimes_{T_X} \mathbb{Q}$. The pushout $\mathbb{Q} \otimes_{T_X} \mathbb{Q}$ lies in $\mathsf{D}(\mathbb{Q})_{\leq 0}$ (for example, compute it by the standard bar construction).

Now we recall the left completion of DTM with respect to $(\mathrm{Ind}(\mathsf{DTM}_{\vee,\geq 0}), \mathrm{Ind}(\mathsf{DTM}_{\vee,\leq 0}))$. In a nutshell, the left completion of DTM is a symmetric monoidal t-exact colimit-preserving functor $\mathsf{DTM}^{\otimes} \to \overline{\mathsf{DTM}}^{\otimes}$ to the "left completed" stable presentable symmetric monoidal ∞-category $\overline{\mathsf{DTM}}^{\otimes}$ (we refer the reader to [25, Section 7.2] and references therein for the notions of left completeness and left completion). The ∞-category $\overline{\mathsf{DTM}}$ can be described as the limit of the diagram indexed by $\mathbb{Z}$

$$\cdots \to \mathsf{DTM}_{\leq n+1} \overset{\tau_{\leq n}}{\to} \mathsf{DTM}_{\leq n} \overset{\tau_{\leq n-1}}{\to} \mathsf{DTM}_{\leq n-1} \overset{\tau_{\leq n-2}}{\to} \cdots$$

of ∞-categories, where $\tau_{\leq n}$ are the truncation functors (we use the homological indexing following [33]). According to [32, 3.3.3] the ∞-category $\overline{\mathsf{DTM}}$ can be identified with the full subcategory of $\mathrm{Fun}(\mathrm{N}(\mathbb{Z}), \mathsf{DTM})$ spanned by functors $\phi : \mathrm{N}(\mathbb{Z}) \to \mathsf{DTM}$ such that

- for any $n \in \mathbb{Z}$, $\phi([n])$ belongs to $\mathsf{DTM}_{\leq -n}$,
- for any $m \leq n \in \mathbb{Z}$, the associated map $\phi([m]) \to \phi([n])$ gives an equivalence $\tau_{\leq -n}\phi([m]) \to \phi([n])$.

Let $\overline{\mathsf{DTM}}_{\geq 0}$ (resp. $\overline{\mathsf{DTM}}_{\leq 0}$) be the full subcategory of $\overline{\mathsf{DTM}}$ spanned by $\phi : \mathrm{N}(\mathbb{Z}) \to \mathsf{DTM}$ such that $\phi([n])$ belongs to $\mathsf{DTM}_{\geq 0}$ (resp. $\mathsf{DTM}_{\leq 0}$) for each $n \in \mathbb{Z}$. The functor $\mathsf{DTM} \to \overline{\mathsf{DTM}}$ induces an equivalence $\mathsf{DTM}_{\leq 0} \to \overline{\mathsf{DTM}}_{\leq 0}$. The pair $(\overline{\mathsf{DTM}}_{\geq 0}, \overline{\mathsf{DTM}}_{\leq 0})$ is an accessible, left complete and right complete t-structure of $\overline{\mathsf{DTM}}$. The functor $\mathsf{DTM} \to \overline{\mathsf{DTM}}$ carries M to $\{\tau_{\leq r} M\}_{r\in\mathbb{Z}}$. Since the t-structure on $\mathsf{D}(\mathbb{Q})$ is left complete, thus the realization functor $\mathsf{DTM}^{\otimes} \to \mathsf{D}^{\otimes}(\mathbb{Q})$ factors as $\mathsf{DTM}^{\otimes} \to \overline{\mathsf{DTM}}^{\otimes} \overset{\overline{\mathsf{R}}_T}{\to} \mathsf{D}^{\otimes}(\mathbb{Q})$ such that $\overline{\mathsf{R}}_T : \overline{\mathsf{DTM}}^{\otimes} \to \mathsf{D}^{\otimes}(\mathbb{Q})$ is conservative by [25, Corollary 7.3].

Return to the proof. Since $\mathsf{DTM}^{\otimes} \to \overline{\mathsf{DTM}}^{\otimes}$ is t-exact, we may and will work with $\overline{\mathsf{DTM}}$ instead of DTM. By abuse of notation, we write $\mathbf{1}_k \otimes_{M_X} \mathbf{1}_k$ for the image in $\overline{\mathsf{DTM}}$. It follows from the conservativity of $\overline{\mathsf{R}}_T$ that $\mathbf{1}_k \otimes_{M_X} \mathbf{1}_k$ belongs to $\overline{\mathsf{DTM}}_{\leq 0}$. Consider the adjunction $\overline{\mathsf{DTM}}_{\geq 0} \rightleftarrows \overline{\mathsf{DTM}} : \tau_{\geq 0}$ where the left adjoint is the symmetric monoidal fully faithful functor. Thus the right adjoint $\tau_{\geq 0} : \overline{\mathsf{DTM}} \to \overline{\mathsf{DTM}}_{\leq 0}$ is lax symmetric monoidal. For any $M \in \mathrm{CAlg}(\overline{\mathsf{DTM}})$, $\tau_{\geq 0}(M)$

is a commutative algebra object in $\overline{\mathsf{DTM}}^{\otimes}_{\geq 0}$. Consequently, $H^0(\mathbf{1}_k \otimes_{M_X} \mathbf{1}_k) = \tau_{\geq 0}(\mathbf{1}_k \otimes_{M_X} \mathbf{1}_k)$ inherits a commutative algebra structure

$$H^0(\mathbf{1}_k \otimes_{M_X} \mathbf{1}_k) \otimes H^0(\mathbf{1}_k \otimes_{M_X} \mathbf{1}_k) \to H^0(\mathbf{1}_k \otimes_{M_X} \mathbf{1}_k), \quad H^0(1_k) \to H^0(\mathbf{1}_k \otimes_{M_X} \mathbf{1}_k)$$

in $\mathrm{Ind}(\mathsf{TM})$. □

We put $M(X,x,y) := H^0(\mathbf{1}_k \otimes_{M_X} \mathbf{1}_k)$. By Proposition A.5 we regard it as a commutative algebra in $\mathrm{Ind}(\mathsf{TM}) \simeq \mathrm{Rep}(MTG)$.

Remark A.6. We can think of $M(X,x,y)$ also as a commutative $\mathbb{Q}$-algebra $H^0(\mathbb{Q} \otimes_{T_X} \mathbb{Q})$ with the canonical action of $MTG \simeq \mathrm{Aut}(R_T)$. This action of MTG on $H^0(\mathbb{Q} \otimes_{T_X} \mathbb{Q})$ can be identified with the action in Section 7, Theorem 7.17. As discussed in Section 7.1, Section 7.4, $\mathsf{MTG} \simeq \mathrm{Aut}(\mathsf{R}_T)$ acts on $\mathbb{Q} \otimes_{T_X} \mathbb{Q} \simeq \mathsf{R}_T(\mathbf{1}_k \otimes_{M_X} \mathbf{1}_k)$. It gives rise to an action of the underlying group scheme MTG on $H^0(\mathbb{Q} \otimes_{T_X} \mathbb{Q})$ (but we treated only the case $x = y$). By [25, Theorem 7.16] and its proof, there is a canonical equivalence $\mathrm{Aut}(\mathsf{R}_T) \simeq \mathrm{Aut}(\overline{\mathsf{R}}_T)$ as functors $\mathrm{CAlg}^{\mathrm{dis}}_{\mathbb{Q}} \to \mathrm{Grp}(\mathcal{S})$ (note that the domain is not $\mathrm{CAlg}_{\mathbb{Q}}$ but $\mathrm{CAlg}^{\mathrm{dis}}_{\mathbb{Q}}$). In addition, by [25, Proposition 7.13, 7.12] $(\overline{\mathsf{DTM}}^{\otimes}, \overline{\mathsf{DTM}}_{\geq 0}, \overline{\mathsf{DTM}}_{\leq 0})$ is a locally dimensional ∞-category in the sense of Lurie [34, VIII, Section 5]. Therefore, the heart is the tannakian category $\mathrm{Rep}^{\otimes}(MTG)$ of (not necessarily finite dimensional) representations of MTG, and the natural morphism $\mathsf{MTG} \to MTG$ in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{dis}}_{\mathbb{Q}}, \mathrm{Grp}(\mathcal{S}))$ can naturally be identified with $\mathrm{Aut}(\mathsf{R}_T) \to \mathrm{Aut}(R_T)$ induced by the restriction of natural equivalences to the heart. Let L be the function field of MTG. Taking account of Theorem 7.17 (2), the action of the group of L-valued point $MTG(L)$ on $H^0(\mathbb{Q} \otimes_{T_X} \mathbb{Q}) \otimes_{\mathbb{Q}} L$ in Theorem 7.17 coincides with the canonical action of $MTG(L) \simeq \mathrm{Aut}(R_T)(L)$ on $R_T(H^0(\mathbf{1}_k \otimes_{M_X} \mathbf{1}_k)) \otimes_{\mathbb{Q}} L \simeq H^0(\mathbb{Q} \otimes_{T_X} \mathbb{Q}) \otimes_{\mathbb{Q}} L$. Since MTG is integral, the coordinate ring on MTG is a subring of L. We then deduce that the action of the group scheme MTG on $H^0(\mathbb{Q} \otimes_{T_X} \mathbb{Q})$ in Theorem 7.17 coincides with the natural action of $MTG \simeq \mathrm{Aut}(R_T)$.

A.4

Theorem A.7. *There is an isomorphism*

$$M_{DG}(X,x,y) \simeq M(X,x,y)$$

in $\mathrm{Ind}(\mathsf{TM})$.

Lemma A.8. *Let* Fin *be the category of (possibly empty) finite sets. Let $\mathcal{C}$ be an ∞-category which has finite coproducts. Let* $\mathrm{Fun}^+(\mathrm{Fin}, \mathcal{C})$ *be the full subcategory of* $\mathrm{Fun}(\mathrm{Fin}, \mathcal{C})$ *spanned by those functors that preserve finite coproducts. Let* $\Delta^0 \to \mathrm{Fin}$ *be the map determined by the set having one element. Then the composition induces an equivalence* $\mathrm{Fun}^+(\mathrm{Fin}, \mathcal{C}) \to \mathrm{Fun}(\Delta^0, \mathcal{C}) = \mathcal{C}$ *of ∞-categories.*

Proof. We here denote by $*$ the set having one element. Since $\mathcal{C}$ has finite coproducts, any functor $\Delta^0 \to \mathcal{C}$ admits a left Kan extension along the inclusion $\Delta^0 = \{*\} \to \mathrm{Fin}$. Moreover, $F : \mathrm{Fin} \to \mathcal{C}$ is a left Kan extension of $F|_{\{*\}}$ if and only if F preserves finite coproducts. Thus, by [32, 4.3.2.15] $\mathrm{Fun}^+(\mathrm{Fin}, \mathcal{C}) \to \mathrm{Fun}(\Delta^0, \mathcal{C}) = \mathcal{C}$ is an equivalence. □

Example A.9. Let $X \in \mathrm{Sm}_k$. Let $\langle X \rangle$ be the subcategory of Sm_k defined as follows: Objects are finite products of X, that is, $\{\mathrm{Spec}\, k, X, X^2, \ldots, X^n, \ldots\}$. A morphism $f : X^n \to X^m$ in Sm_k is a morphism in $\langle X \rangle$ if and only if f is of the form $X^n \to X^m$, $(x_1, \ldots x_n) \mapsto (x_{i_1}, \ldots x_{i_m})$ for some $\{i_1, \ldots, i_m\} \subset \{1, \ldots, n\}$. Then there is an equivalence $\langle X \rangle^{op} \simeq \mathrm{Fin}$ which carries X^n to the set having n elements.

Proof of Theorem A.7. We will prove that there is a natural isomorphism $M_{DG}(X, x, y) \simeq M(X, x, y)$ in $\mathrm{Ind}(\mathsf{TM})$. Note the equivalence $\mathbf{1}_k \otimes_{M_X} \mathbf{1}_k \simeq \mathbf{1}_k \otimes_{M_X \otimes M_X} M_X$ in $\mathrm{CAlg}(\mathsf{DTM}_\vee)$ where the right hand side is determined by $x^* \otimes y^* : M_X \otimes M_X \to \mathbf{1}_k \otimes \mathbf{1}_k \simeq \mathbf{1}_k$ and $M_X \otimes M_X \simeq M_{X \times X} \to M_X$ induced by the diagonal $X \to X \times X$. Here the two projections $X \leftarrow X \times X \to X$ determines a canonical equivalence $M_X \otimes M_X \to M_{X \times X}$ in $\mathrm{CAlg}(\mathsf{DTM}_\vee^\otimes)$ (one way to see this is to observe that the conservative realization $\mathrm{CAlg}(\mathsf{DTM}_\vee^\otimes) \to \mathrm{CAlg}_{\mathbb{Q}}$ sends $M_X \otimes M_X \to M_{X \times X}$ to $T_X \otimes T_X \to T_{X \times X}$ that is an equivalence by Künneth formula). Next we define a certain "resolution" of M_X over $M_X \otimes M_X$. For this purpose, let us consider the following cosimplicial scheme

$$R^\Delta(X) : \Delta \to \mathrm{Sm}_k, \quad [n] \mapsto X' \times X^n \times X''$$

over $X' \times X'' = X \times X$. Here, to avoid confusion we put $X' = X$ and $X'' = X$, and $X' \times X''$ is regarded as the constant cosimplicial scheme. Cofaces are given by

$$d^i(x_0, x_1, \ldots, x_{n+1}) = (x_0, \ldots, x_{n-i+1}, x_{n-i+1}, \ldots x_{n+1}), \quad 0 \le i \le n+1,$$

and codegeneracies are defined by projections. If $X \to X' \times X''$ is the diagonal morphism, then $R^\Delta(X)$ has a coaugmentation $X \to R^\Delta(X)$ over $X' \times X''$. Observe that there is a the fiber product of cosimplicial schemes

$$\begin{array}{ccc} P^\Delta(X, x, y) & \longrightarrow & R^\Delta(X) \\ \downarrow & & \downarrow \\ \mathrm{Spec}\, k = (y, x) & \longrightarrow & X' \times X'' \end{array}$$

where the right vertical map is the projection, and $\mathrm{Spec}\, k$ is considered to be the constant cosimplicial scheme. For each cosimplicial scheme, composing it with $\Xi : \mathrm{Sm}_k^{op} \to \mathrm{CAlg}(\mathsf{DM}^\otimes(k))$ we obtain simplicial objects $\mathcal{M}_\Delta(X, x, y)$, $\mathcal{M}_\Delta(X)$, $M_{X'} \otimes M_{X''}$, $\mathbf{1}_k$ in $\mathrm{CAlg}(\mathsf{DM}^\otimes(k))$ respectively from $P^\Delta(X, x, y)$, $R^\Delta(X)$, $X' \times X''$ and $\mathrm{Spec}\, k$. Each term of these simplicial objects lies in $\mathrm{CAlg}(\mathsf{DTM}_\vee)$ since $M_{X^n} \simeq M_X^{\otimes n}$. Consider the pushout $\mathbf{1}_k \otimes_{M_{X'} \otimes M_{X''}} \mathcal{M}_\Delta(X)$ of simplicial objects (which consists of termwise pushouts). There is a natural morphism of simplicial objects

$$\mathbf{1}_k \otimes_{M_{X'} \otimes M_{X''}} \mathcal{M}_\Delta(X) \to \mathcal{M}_\Delta(X, x, y).$$

This morphism is an equivalence. To see this, it will suffice to prove that the morphism in each term is an equivalence. The morphism in the n-th term is equivalent to

$$\mathbf{1}_k \otimes_{M_{X'} \otimes M_{X''}} M_{X'} \otimes M_{X^n} \otimes M_{X''} \to M_{\{y\} \times X^n \times \{x\}}$$

which is an equivalence. Let $\mathcal{M}(X)$ be a colimit of $\mathcal{M}_\Delta(X)$ in $\mathrm{CAlg}(\mathsf{DTM})$. The coaugmentation $X \to R^\Delta(X)$ over $X' \times X''$ gives rise to $\mathcal{M}(X) \to M_X$ over $M_{X'} \otimes M_{X''}$. Since $M_{DG}(X, x, y) = H^0(\mathcal{M}(X, x, y))$, we will show that the induced map

$$H^0(\mathbf{1}_k \otimes_{M_{X'} \otimes M_{X''}} \mathcal{M}(X)) \to H^0(\mathbf{1}_k \otimes_{M_X \otimes M_X} M_X)$$

is an isomorphism in Ind(TM). To this end, recall the left completion $\mathsf{DTM}^{\otimes} \to \overline{\mathsf{DTM}}^{\otimes}$ from the second paragraph of the proof of Proposition A.5. It is symmetric monoidal, t-exact and colimit-preserving. We may and will replace $\mathsf{DTM}^{\otimes}$ by $\overline{\mathsf{DTM}}^{\otimes}$. We show that $\mathbf{1}_k \otimes_{M_{X'} \otimes M_{X''}} M_X$ is the colimit of $\mathbf{1}_k \otimes_{M_{X'} \otimes M_{X''}} \mathcal{M}_{\Delta}(X)$ in $\overline{\mathsf{DTM}}$. The image of $\mathcal{M}_{\Delta}(X)$ under the realization functor is the simplicial diagram in $\mathrm{CAlg}_{\mathbb{Q}}$ given by the composite

$$s : \Delta^{op} \overset{R^{\Delta}(X)^{op}}{\to} \mathrm{Sm}_k^{op} \overset{\Xi}{\to} \mathrm{CAlg}(\mathsf{DM}^{\otimes}(k)) \overset{\mathsf{R}}{\to} \mathrm{CAlg}_{\mathbb{Q}}, \quad [n] \mapsto T_{X' \times X^n \times X''}.$$

Since $\mathrm{CAlg}(\overline{\mathsf{DTM}}) \overset{\overline{\mathsf{R}}_T}{\to} \mathrm{CAlg}_{\mathbb{Q}}$ is conservative and colimit-preserving, we are reduced to proving that $s : [n] \mapsto T_{X' \times X^n \times X''}$ in $\mathrm{CAlg}_{\mathbb{Q}}$ has a colimit T_X. We let $F_X : \langle X \rangle^{op} \to \mathrm{CAlg}_{\mathbb{Q}}$ be the functor given by $X^m \mapsto T_{X^m}$. The natural projections induce $T_X^{\otimes m} = T_X \otimes \ldots \otimes T_X \overset{\sim}{\to} T_{X^m}$, and $T_{\mathrm{Spec}\, k} \simeq \mathbb{Q}$. By Lemma A.8 and Example A.9, there is a canonical equivalence $\mathrm{Fun}^{+}(\langle X \rangle^{op}, \mathrm{CAlg}_{\mathbb{Q}}) \simeq \mathrm{CAlg}_{\mathbb{Q}}$ which carries F to $F(X)$. Since F_X belongs to $\mathrm{Fun}^{+}(\langle X \rangle^{op}, \mathrm{CAlg}_{\mathbb{Q}})$, the functor F_X that preserves finite coproducts is "uniquely determined" by $F_X(X) = T_X$. Let A be a cofibrant commutative dg algebra over $\mathbb{Q}$ that represents T_X. Let $\mathrm{CAlg}_{\mathbb{Q}}^{dg} \to \mathrm{CAlg}_{\mathbb{Q}}$ be the canonical functor (see Section 2). Let $f_A : \langle X \rangle^{op} \to \mathrm{CAlg}_{\mathbb{Q}}^{dg}$ be the functor given by $X^m \mapsto A^{\otimes m}$, which corresponds to A through the canonical equivalence $\mathrm{Fun}^{+}(\langle X \rangle^{op}, \mathrm{CAlg}_{\mathbb{Q}}^{dg}) \simeq \mathrm{CAlg}_{\mathbb{Q}}^{dg}$. The composite $F_A : \langle X \rangle^{op} \to \mathrm{CAlg}_{\mathbb{Q}}$ is the functor that preserves finite coproducts. Thus $F_A \in \mathrm{Fun}^{+}(\langle X \rangle^{op}, \mathrm{CAlg}_{\mathbb{Q}})$. It follows from $A \simeq T_X$ in $\mathrm{CAlg}_{\mathbb{Q}}$ that $F_A \simeq F_X$. Note that $R^{\Delta}(X)^{op} : \Delta^{op} \to \mathrm{Sm}_k^{op}$ uniquely factors through the subcategory $\langle X \rangle^{op} \to \mathrm{Sm}_k^{op}$. The composite $s : \Delta^{op} \to \langle X \rangle^{op} \overset{F_X}{\to} \mathrm{CAlg}_{\mathbb{Q}}$ is equivalent to $s' : \Delta^{op} \to \langle X \rangle^{op} \overset{F_A}{\to} \mathrm{CAlg}_{\mathbb{Q}}$. We may replace s by s'. By unfolding the definition, the simplicial commutative dg algebra $s' : \Delta^{op} \to \mathrm{CAlg}_{\mathbb{Q}}^{dg}$, $[n] \mapsto A \otimes A^{\otimes n} \otimes A$ (over $A \otimes A$) is the simplicial bar resolution of A over $A \otimes A$: $[n] \mapsto A \otimes A^{\otimes n} \otimes A$ (see [41, 4.3, 4.4, 4.6] or [51, 3.7] for what this means). The (homotopy) colimit of the simplicial bar resolution $[n] \mapsto A \otimes A^{\otimes n} \otimes A$ (equivalently the totalization) is naturally equivalent to A. (We remark that a colimit of a simplicial diagram of commutative algebra objects is a colimit of simplicial diagram of underlying objects.) Consequently, $\mathbf{1}_k \otimes_{M_{X'} \otimes M_{X''}} \mathcal{M}(X) \simeq \mathbf{1}_k \otimes_{M_{X'} \otimes M_{X''}} M_X$ in $\overline{\mathsf{DTM}}$. Hence we obtain a canonical isomorphism $M_{DG}(X, x, y) \simeq M(X, x, y)$ in Ind(TM). □

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