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HIGHER STRUCTURES

Platonic and alternating 2-groups

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Abstract

We recall Schur's work on universal central extensions and develop the analogous theory for categorical extensions of groups. We prove that the String 2-groups are universal in this sense and proceed study their restrictions to the finite subgroups of the 3-sphere $Spin(3)$ and to the spin double covers of the alternating groups. We find that almost all of these restrictions are universal categorical extensions and that the categorical extensions of the alternating family are governed by the stable 3-stem $\pi_3(\mathbb{S}^0)$.

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1. Introduction

This paper is devoted to the study of the string 2-groups $String(n)$, after restriction to two classes of finite subgroups of the groups $Spin(n)$. The first class consists of the finite subgroups of the 3-sphere

$$\mathbb{S}^3 = SU(2) = Spin(3).$$

Their list consists of the cyclic and the binary dihedral groups, plus the three exceptional cases: the binary tetrahedal group $2T \cong \widetilde{A}_4$, the binary octahedral group $2O \cong \widetilde{S}_4$, and the binary icosahedral group $2I \cong \widetilde{A}_5$. Let $\boldsymbol{\mu}_n \subset U(1)$ denote the group of complex nth roots of unity. Then the cocycle classifying the restriction $String(3)|_G$ is cohomologous to one with values in $\boldsymbol{\mu}_{|G|}$ and exact order $|G|$. This means that we have a categorical extension $\mathcal{G}$ of G by $\mathbb{B}\boldsymbol{\mu}_{|G|}$, sitting inside $String(3)$ as follows

$$\begin{array}{ccccc} \mathbb{B}\boldsymbol{\mu}_{|G|} & \longrightarrow & \mathcal{G} & \longrightarrow & G \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ \mathbb{B}U(1) & \longrightarrow & String(3) & \longrightarrow & \mathbb{S}^3. \end{array}$$

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We will refer to such $\mathcal{G}$ as *platonic 2-groups*. These are interesting objects, because they satisfy a (weak) universal property, suggesting a categorical aspect of McKay correspondence that seems worth exploring. The second family of examples consists of the *alternating 2-groups* $\mathcal{A}_n$, which are related to the stable homotopy groups of spheres by the tower

$$\begin{array}{ccc} A_n & \xhookrightarrow{\bar{\varrho}_n} & SO(n) \\ \downarrow & & \downarrow \\ S_n & \xhookrightarrow{\varrho_n} & O(n) \\ \downarrow & & \downarrow \\ \pi_1(\mathbb{S}^0) & = & \pi_0(O) \end{array} \qquad \begin{array}{ccc} \pi_2(\mathbb{S}^0) & = & \pi_1(O) \\ \downarrow & & \downarrow \\ \widetilde{A}_n & \xhookrightarrow{\widetilde{\varrho}_n} & Spin(n) \\ \downarrow & & \downarrow \\ A_n & \xhookrightarrow{\bar{\varrho}_n} & SO(n) \end{array} \qquad \begin{array}{ccc} \mathbb{B}\pi_3(\mathbb{S}^0) & \xhookrightarrow{e} & \mathbb{B}U(1) \\ \downarrow & & \downarrow \\ \mathscr{A}_n & \hookrightarrow & String(n) \\ \downarrow & & \downarrow \\ \widetilde{A}_n & \xhookrightarrow{\widetilde{\varrho}_n} & Spin(n). \end{array}$$

Here

$$O = colim O(n)$$

is the infinite orthogonal group, and the homotopy groups turning up are

$$\pi_1(\mathbb{S}^0) \cong \boldsymbol{\mu}_2, \qquad \pi_2(\mathbb{S}^0) \cong \boldsymbol{\mu}_2, \qquad \pi_3(\mathbb{S}^0) \cong \boldsymbol{\mu}_{24}, \qquad \text{and} \qquad B\pi_3(O) = U(1).$$

The homomorphism $\widetilde{\varrho}_n$ is the permutation representation, and

$$e\colon \pi_3(\mathbb{S}^0) \longrightarrow U(1)$$

is the Adams e-invariant. In the philosophy of [Kap15], the stable 1-stem yields the sign governing super-symmetry, while the stable 2-stem provides the sign governing categorified super-symmetry. It was Kapranov's question about a conceptual description of the stable 3-stem in this context that motivated our work. For n sufficiently large, the alternating 2-groups become universal categorical central extensions. A consequence is the following.

Theorem 1.1. *The restriction of $String(n)$ to $\widetilde{A}_n$ has exact order 24 for all $n \geq 4$.*

As above, "order" refers to the order of the classifying cocycle. It would be interesting to have a direct proof of Theorem 1.1, using any of the known constructions of the String 2-groups. Further, one can think of $\pi_3(\mathbb{S}^0)$ as framed bordism group, generated by the three sphere in its invariant framing and use a $K3$-surface with little holes cut out as a null-bordism of $24[\mathbb{S}^3]$. This suggests a potential connection with the categorical groups turning up in Mathieu Moonshine.

2. Categorified Schur theory

2.1 Basic definitions and notation Throughout, G, $\widetilde{G}$ and A denote finite dimensional Lie groups, A is abelian, and $\mathbb{B}A$ denotes the single object (monoidal) Lie groupoid with automorphism group A, and similarly $\mathbb{B}^2A$ is such a 2-groupoid. We write $H^*_{gp}(G; A)$ for the Lie group cohomology of G with coefficients in A as axiomatically characterized in [WW15]. In low degrees, this has the following meaning.[1]

1. The first group consists of the homomorphisms

$$H^1_{gp}(G; A) \cong Hom(G, A).$$

[1] What follows is a summary of results of Schur (1911), Singh (1976), Schommer-Pries [SP11, Thm.99], Wagemann and Wockel [WW15], Schreiber [Sch13].

2. Let $Cent(G, A)$ be the groupoid of central extensions

$$1 \longrightarrow A \longrightarrow \widetilde{G} \longrightarrow G \longrightarrow 1.$$

Then $Cent(G, A)$ is equivalent to the skeletal groupoid

$$H^2_{gp}(G; A) \times \mathbb{B}H^1_{gp}(G; A).$$

3. Let $\mathcal{C}ent(G, \mathbb{B}A)$ be the bicategory of central extensions of categorical Lie groups

$$1 \longrightarrow \mathbb{B}A \longrightarrow \mathcal{G} \longrightarrow G \longrightarrow 1$$

as in [SP11]. Then $\mathcal{C}ent(G, \mathbb{B}A)$ is equivalent to the skeletal bicategory

$$H^3_{gp}(G; A) \times \mathbb{B}H^2_{gp}(G; A) \times \mathbb{B}^2 H^1_{gp}(G; A).$$

When G is discrete, a categorical extension as in (3) consists of a monoidal groupoid $(\mathcal{G}, \bullet, 1)$ with weakly invertible objects, together with isomorphisms[2]

$$G \;\cong\; ob(\mathcal{G})/\cong$$

and

$$A \;\cong\; Aut_{\mathcal{G}}(1).$$

Such an extension is central, if for each $a \in A$ and $g \in ob(\mathcal{G})$, we have

$$a \bullet g = g \bullet a$$

in $Aut_{\mathcal{G}}(g)$. For discrete G, the definition of Lie group cohomology specializes to that of classical group cohomology and is computed using the familiar bar construction. Let

$$\alpha : G \times G \times G \longrightarrow A,$$

be a normalized 3-cocycle with values in some G-module A. Then α classifies the skeletal categorical group

$$\mathcal{G}_\alpha = \begin{pmatrix} G \ltimes A \\ \downarrow\downarrow \\ G \end{pmatrix}$$

(source and target map are projection to the first factor). The monoidal structure on $\mathcal{G}_\alpha$ consists of the multiplications in G, respectively $G \ltimes A$ with associators encoded in α and trivial unit maps. This is a categorical extension of G by $\mathbb{B}A$, which is central if and only if A is trivial as G-module. For discrete G, we may drop the subscript 'gp' for group cohomology, which in this case is isomorphic to the singular cohomology of the classifying space

$$H^*_{gp}(G; A) \;\cong\; H^*(BG; A).$$

[2] A common notation is to write $\pi_0(\mathcal{G}) = ob(\mathcal{G})/\cong$ for the isomorphism classes of objects and $\pi_1(\mathcal{G}) = Aut_{\mathcal{G}}(1)$ for the Bernstein center of $\mathcal{G}$. Due to the importance of the stable stems below, we will avoid this notation.

This is related to (integral) group homology

$$H_*(G) = H_*(BG, \mathbb{Z})$$

via the universal coefficient theorem

$$0 \longrightarrow Ext(H_{i-1}(G), A) \longrightarrow H^i(G; A) \longrightarrow Hom(H_i(G), A) \longrightarrow 0. \quad (4)$$

We note that (4) is natural in A. In general, there is no definition of Lie group homology. For this reason, our approach to Schur theory will be rather different in the discrete versus the compact connected case. We recall one more construction, which, in a sense, is the opposite extreme to the skeletal 2-groups discussed above.

Definition 2.1. A strict categorical Lie group is a group object in the category of finite dimensional Lie groupoids.

This is interpreted as a categorical Lie group with identity arrows as associators and units. The data of a strict categorical Lie group $\mathcal{G}$ are equivalent to those of the crossed module

$$t\colon ker(s) \longrightarrow ob(\mathcal{G}),$$

where s and t are the source maps of $\mathcal{G}$ and target and $ob(\mathcal{G})$ acts on $ker(s)$ by conjugation. A strict categorical group has identities as units and associators. In particular, a skeletal categorical group is strict if and only if it is trivial. In Section 4, we discuss an example of two equivalent categorical groups, one strict and the other skeletal.

2.2 Schur Theory Recall that the first homology group of a discrete group is its abelianization,

$$H_1(G) \cong G^{ab}$$

and that $H_2(G)$ is the *Schur multiplier* of G. We will refer to $H_3(G)$ as the *categorical Schur multiplier* of G. Recall further that a group G is called *perfect* if its abelianization is trivial and that a perfect group is called *superperfect* if its Schur multiplier also vanishes. The smallest non-trival example of a superperfect group is the binary icosahedral group, whose categorical Schur multiplier is

$$H_3(2I) \cong \boldsymbol{\mu}_{120},$$

see [Hau78]. A list of the categorical Schur multipliers of some superperfect groups exists as HAP library.

We now fix the Lie group G, but allow the centre A to vary. Write $Cent(G)$ for the category of finite dimensional central extensions of G and $\mathcal{C}ent(G)$ for the bicategory of finite dimensional categorical central extensions of the form (3).

Definition 2.2. A *Schur cover* of G is an initial object of $Cent(G)$. A *categorical Schur cover* of G is an initial object of $\mathcal{C}ent(G)$.

If it exists, the Schur cover of G is unique up to unique isomorphism. Similarly, the categorical Schur cover of G, if it exists, is unique up to equivalence, which in turn is unique up to unique isomorphism. We recall two classical results about Schur covers.

Theorem 2.3 (Schur 1904)**.** *Let G be a perfect discrete group. Then G possesses a Schur cover, whose central subgroup is the Schur multiplier*

$$A_{uni} \;=\; H_2(G).$$

Lemma 2.4 (Second Whitehead Lemma)**.** *Let G be a semisimple, compact and connected Lie group. Then the universal covering group of G is a Schur cover. Its central subgroup is the fundamental group*

$$A_{uni} \;=\; \pi_1(G).$$

To ephasize the analogy between these two statements, let BG be the classifying space of G. Then we have

$$\pi_i(BG) \;=\; \pi_{i-1}(G).$$

So, the Lie group G is connected if and only if $\pi_1(BG)$ is trivial. In this case, we have

$$H_1(BG;\mathbb{Z}) \;=\; 0$$

and

$$H_2(BG;\mathbb{Z}) \;=\; \pi_2(BG) \;\cong\; \pi_1(G).$$

The goal of this section is to prove the following result.

Theorem 2.5.

1. *Let G be a superperfect discrete group. Then G possesses a categorical Schur cover, whose center is*
$$A_{uni} = H_3(G).$$
2. *Let G be a simply connected compact Lie group, and let s be the number of simple factors of G. Then G possesses a categorical Schur cover, whose centre is*
$$A_{uni} = U(1)^s.$$

Note that simply connected compact Lie groups are automatically semi-simple [MT91, Thm.5.29], so that the statement in (2) makes sense.

2.3 The theory for discrete G

Given an abelian group H, we will write $H \downarrow \mathcal{A}b$ for the category of abelian groups under H.

Proof of Theorems 2.3 and 2.5 (1). Assume that G is perfect, and let A be an abelian group, viewed as trivial G-module. Then the universal coefficient theorem with $i = 1$ implies

$$0 \;=\; H^1(G;A).$$

It follows that the groupoid of central extensions of G by A possesses no non-identity automorphisms. Using the universal coefficient theorem with $i = 2$, it follows that the isomorphism classes of said groupoid are parametrised by $Hom(H_2(G), A)$. Now allow A to vary. Then we obtain an equivalence from the category of central extensions of G to the under category $H_2(G) \downarrow \mathcal{A}b$, sending a central extension to the homomorphism classifying it. In particular, there is a universal central extension, which is characterised uniquely, up to unique isomorphism, by the fact that it is classified by $id_{H_2(G)}$. The proof for Theorem 2.5 (1) is analogous. □

We can be more precise about the equivalence of the proof.

Definition 2.6. Assume we are given a categorical central extension

$$1 \longrightarrow \mathbb{B}A \longrightarrow \mathcal{G} \longrightarrow G \longrightarrow 1.$$

and a homomorphism $\phi\colon A \to B$ to another abelian group. Then the *categorical group with center B associated to $\mathcal{G}$* (via ϕ) is the groupoid $\mathcal{G}[\phi]$ with objects identical to those of $\mathcal{G}$ and arrows the pairs (f, b) with f an arrow of $\mathcal{G}$ and $b \in B$, modulo the equivalence relation

$$(f \bullet a, b) \sim (f, \phi(a) + b), \qquad a \in A.$$

The multiplication data are inherited from $\mathcal{G}$.

Theorem 2.7. *1. For a perfect group G with Schur cover $\widetilde{G}_{uni}$, the functor*

$$\widetilde{G}_{uni}[-] : H_2(G)\!\downarrow \mathcal{A}b \longrightarrow \mathcal{C}ent(G)$$

sending the homomorphism

$$\phi : H_2(G) \longrightarrow A$$

to the balanced product

$$A \times_{H_2(G)} \widetilde{G}_{uni}$$

is an equivalence of categories.

2. For a superperfect group G with categorical Schur cover $\mathcal{G}_{uni}$, the functor

$$\mathcal{G}_{uni}[-] : H_3(G)\!\downarrow \mathcal{A}b \longrightarrow \mathcal{C}ent(G)$$

is an equivalence of (bi)categories.

Proof. The fisrt part is classical, we prove (2). The universal coefficient theorem implies that the bicategory $\mathcal{C}ent(G)$ has only identity 2-morphisms and that we have an abstract equivalence

$$H_3(G)\!\downarrow \mathcal{A}b \xrightarrow{\ \sim\ } \mathcal{C}ent(G).$$

The identity map of $H_3(G)$ is an initial object of $H_3(G)\!\downarrow \mathcal{A}b$. Under the isomorphism of the universal coefficient theorem, this corresponds to the class of a 3-cocycle α_{uni} with values in $H_3(G)$, and for arbitrary A, the universal coefficient isomorphism is

$$\begin{array}{ccc} H^3(G;A) & \cong & Hom(H_3(G), A) \\ [\phi_* \, \alpha_{uni}] & \longmapsto & \phi, \end{array}$$

by naturality. If

$$\mathcal{G}_{uni} = \mathcal{G}_{\alpha_{uni}},$$

then, by construction,

$$\mathcal{G}_{\phi_* \alpha_{uni}} \simeq \mathcal{G}_{uni}[\phi].$$

An inverse of the functor $\mathcal{G}_{uni}[-]$ restricts the unique 1-morphism $\mathcal{G}_{uni} \longrightarrow \mathcal{G}$ to centers. □

Let G be a discrete group, not necessarily perfect. Assume that the Schur multiplier $H_2(G)$ vanishes. Then we still have the cohomology class $[\alpha_{uni}]$ and $\mathcal{G}_{uni}$ as in the above proof.

Definition 2.8. We will refer to any Lie 2-group extension equivalent to $\mathcal{G}_{uni}$ as a *weak categorical Schur cover* of G.

In the situation of the definition,

$$\mathcal{G}_{uni}[-] : H_3(G)\downarrow \mathcal{A}b \longrightarrow \mathcal{C}ent(G)$$

is still essentially bijective, but may no longer be an equivalence of bicategories. We conclude our discussion of the discrete case by recalling the following corollary of the universal coefficient theorem.

Corollary 2.9. *If G is finite, we have a non-canonical isomorphism*

$$H^i(G;U(1)) \xrightarrow{\cong} \widehat{H_i(G)} \cong H_i(G).$$

In particular, the (categorical) Schur multiplier of a finite group G classifies (categorical) central extensions of G by the circle group.

Proof. The circle group is injective. □

The categorical central extensions by the circle group are of interest for the theory of projective 2-representations [GU16], just like central group extensions by the circle group are of interest for the theory of projective representations.

2.4 The theory for compact connected G Because of the second Whitehead Lemma, the role played by *superperfect* discrete groups is played by the *simply connected* compact Lie groups. Let G be a simply connected compact Lie group, and let T be a finite dimensional abelian Lie group with cocharacter lattice

$$\check{T} := Hom(U(1), T).$$

Theorem 2.10. *Let s be the number of simple factors of G. Then*

$$H^1_{gp}(G;T) = H^2_{gp}(G;T) = 0,$$

and we have an isomorphism

$$H^3_{gp}(G;T) \cong \check{T}^s,$$

which is natural in G.

Proof. We have

$$\pi_i(BG) = \pi_{i-1}(G) = \begin{cases} 0 & i \leq 3 \\ \mathbb{Z}^s & i = 4, \end{cases}$$

[MT91, Thm. 4.17]. Using Hurewitz and the universal coefficient theorem, this implies

$$H^i_{gp}(G;A) = H^i(BG;A) = \begin{cases} 0 & 1 \leq i \leq 3 \\ A^s & i = 4 \end{cases}$$

for discrete coefficients A. If T_0 is the connected component of 0, then the short exact sequence

$$T_0 \hookrightarrow T \twoheadrightarrow T/T_0,$$

gives isomorphisms

$$H^i_{gp}(G;T_0) \;\cong\; H^i_{gp}(G;T)$$

for $1 \le i \le 3$. Let $S \subseteq T_0$ be a maximal compact subgroup. Then T_0 is the product of S with $\mathbb{R}^m$ for some m. By [Hu52, Thm. 2.8], the cohomology of a compact Lie group with coefficients in $\mathbb{R}^m$ vanishes for $i \ge 1$. Hence the short exact sequence

$$S \hookrightarrow T_0 \twoheadrightarrow T_0/S$$

gives isomorphisms

$$H^i_{gp}(G;S) \;\cong\; H^i_{gp}(G;T_0).$$

At the same time, we have

$$\check{S} \;=\; \check{T}.$$

We may therefore assume, without loss of generality, that T is a compact torus. Finally, the long exact cohomology sequence for

$$\check{T} \hookrightarrow \mathfrak{t} \twoheadrightarrow T,$$

with $\mathfrak{t}$ the Lie algebra of T, gives isomorphisms

$$H^i_{gp}(G;T) \;\cong\; H^{i+1}(BG;\check{T}).$$

□

The statements of Lemma 2.4 and Theorem 2.5 (2) can be derived from this result:

Proof of the Second Whitehead Lemma. By Weyl's theorem, the simply connected group $\widetilde{G}$ is again compact, and by Theorem 2.10, it has no non-trivial central extension with finite dimensional center. □

Proof of Theorem 2.5 (2). We have functorial isomorphisms

$$\begin{aligned}\check{T}^s &= Hom(\mathbb{Z}^s, Hom(U(1),T))\\ &\cong Hom(U(1)\otimes\mathbb{Z}^s, T)\\ &= Hom(U(1)^s, T).\end{aligned}$$

So, Theorem 2.10 implies that the bicategory $\mathcal{C}ent(G)$ is equivalent to the category of finite dimensional abelian Lie groups under $U(1)^s$. □

Example 2.11 ([SP11, Thm.100]). For $n = 3$ and $n \ge 5$, the string extension

$$\mathbb{B}U(1) \longrightarrow String(n) \longrightarrow Spin(n)$$

is the universal central Lie 2-group extension of the simple and simply connected Lie group $Spin(n)$.

3. The string cocycle

This section summarizes the restriction of the construction of $String(n)$ in [Woc11] and [WW15] to a finite subgroup of $Spin(n)$. Following [BtD95], we identify the maximal torus of $Spin(n)$ with

$$T = \mathbb{R}^{\lfloor \frac{n}{2} \rfloor}/\mathbb{Z}_{ev}^{\lfloor \frac{n}{2} \rfloor},$$

where

$$\mathfrak{t} = \mathbb{R}^{\lfloor \frac{n}{2} \rfloor}$$

is the Lie algebra and

$$\Lambda^\vee = \mathbb{Z}_{ev}^{\lfloor \frac{n}{2} \rfloor} = \{m \in \mathbb{Z}^{\lfloor \frac{n}{2} \rfloor} \mid \langle m, m \rangle\}$$

is the coweight lattice. The basic bilinear form $\langle -, - \rangle$ on $\mathfrak{spin}(\mathfrak{n})$ is then the multiple of the Killing form that restricts to the standard scalar product on $\mathbb{R}^{\lfloor \frac{n}{2} \rfloor}$. The Cartan three form is the invariant three form ν on $Spin(n)$ with

$$\nu_1(\xi, \zeta, \eta) = \langle [\xi, \zeta], \eta \rangle.$$

Restricted to $\mathbb{S}^3 = Spin(3)$, we have

$$\nu_1(\xi, \zeta, \eta) = \langle \xi \times \zeta, \eta \rangle = \det(\xi, \zeta, \eta).$$

So, ν is the volume form. Let now $G \subset Spin(n)$ be a finite subgroup, and let

$$\mathbb{Z} \longleftarrow Bar_\bullet G$$

be the bar resolution,

$$Bar_k G = \mathbb{Z}[G]G^k.$$

Let $C_\bullet(Spin(n))$ be the singular chain complex of $Spin(n)$. Since $Spin(n)$ is 2-connected, and $Bar_\bullet G$ is free, we may choose maps

$$f_i \colon Bar_i G \longrightarrow C_i(Spin(n)),$$

for $0 \le i \le 3$, such that f_0 maps $g()$ to the 0-simplex g in $Spin(n)$, and the f_i fit together to form a map of truncated chain complexes of $\mathbb{Z}[G]$-modules. Here G acts on the simplices in $Spin(n)$ by left translation.

Explicitly, a choice of f amounts to, for each $g \in G$, a path γ_g from 1 to g, for each pair $(g|h)$ of elements of G, a 2-simplex $\Delta_{g,h}$ bounding

$$\gamma_g - \gamma_{gh} + g\gamma_h,$$

and for each triple $(g|h|k)$, a 3-simplex $W_{g,h,k}$ bounding

$$-\Delta_{g,h} + \Delta_{g,hk} - \Delta_{gh,k} + g\Delta_{h,k}.$$

Definition 3.1. For a fixed choice of $f_\bullet$, let

$$\alpha \colon G^3 \longrightarrow \mathbb{R}/\mathbb{Z}$$

be the 3-cocycle

$$\alpha(g|h|k) = \frac{1}{2\pi^2} \int_{W_{g,h,k}} \nu \qquad \text{mod } \mathbb{Z}.$$

Lemma 3.2. *Different choices of $f_\bullet$ yield cohomologous choices of α.*

Proof. Let $f'_\bullet$ be a second choice for $f_\bullet$, and let α' be the resulting 3-cocycle. Employing again the 2-connectedness of $Spin(n)$, we obtain a chain homotopy

$$\begin{array}{ccccccc}
Bar_0 G & \xleftarrow{\ \delta\ } & Bar_1 G & \xleftarrow{\ \delta\ } & Bar_2 G & \xleftarrow{\ \delta\ } & Bar_3 G \\
\downarrow 0 & \searrow^{0} & \downarrow f_1 - f'_1 & \searrow^{H_1} & \downarrow f_2 - f'_2 & \searrow^{H_2} & \downarrow f_3 - f'_3 \\
C_0(Spin(n)) & \xleftarrow[\delta]{} & C_1(Spin(n)) & \xleftarrow[\delta]{} & C_2(Spin(n)) & \xleftarrow[\delta]{} & C_3(Spin(n))
\end{array}$$

relating $f_\bullet$ and $f'_\bullet$ up to degree 2 and such that

$$f_3 - f'_3 - H_2 \circ \delta$$

takes values in the 3-cycles $Z_3(Spin(n))$. Letting β be the 2-cocycle

$$\beta(g|h) \;=\; \frac{1}{2\pi^2} \int_{H_2(g|h)} \nu \qquad \bmod \mathbb{Z},$$

it follows that

$$\alpha - \alpha' \;=\; \delta^* \beta.$$

□

4. The cyclic groups

As a warm-up to the platonic and alternating case, we study the categorical extensions of the finite subgroups of the circle group. The finite cyclic groups have integral homology

$$H_i(\boldsymbol{\mu}_n) \;=\; \begin{cases} \mathbb{Z} & i = 0, \\ \boldsymbol{\mu}_n & i \text{ odd, and} \\ 0 & \text{else.} \end{cases}$$

This implies that $\boldsymbol{\mu}_n$ possesses a weak categorical Schur cover $\mathcal{C}_n$, whose centre is $\boldsymbol{\mu}_n$. Let $\mathcal{U}(1)^-$ be the categorical extension of the circle group classified by the standard generator[3] of

$$H^3_{gp}(U(1); U(1)) \;\cong\; H^4(BU(1); \mathbb{Z}) \;\cong\; \mathbb{Z}.$$

We will see that there is a 1-morphism of categorical central extensions

$$\begin{array}{ccccc}
\mathbb{B}\boldsymbol{\mu}_n & \longrightarrow & \mathcal{C}_n & \longrightarrow & \boldsymbol{\mu}_n \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{B}U(1) & \longrightarrow & \mathcal{U}(1)^- & \longrightarrow & U(1),
\end{array}$$

identifying $\mathcal{C}_n$ with a sub-categorical group of $\mathcal{U}(1)^-$. Let $\mathbb{R}$ act on $\mathbb{Z} \times U(1)$ by

$$x \cdot (m, z) \;:=\; \left(m, z \cdot e^{-2\pi i m x}\right),$$

and recall that

[3] In [Gan14], we make the convention that the basic categorical extension of the circle group is the 2-group $\mathcal{U}(1)$ classified by the other generator. These two 2-groups differ by a sign in the action.

$$\mathcal{U}(1)^- = \begin{pmatrix} \mathbb{R} \ltimes (\mathbb{Z} \times U(1)) \\ \downarrow \downarrow \\ \mathbb{R} \end{pmatrix}$$

is constructed as the strict categorical group corresponding to the crossed module

$$\begin{aligned} \boldsymbol{\upsilon} : \mathbb{Z} \times U(1) &\longrightarrow \mathbb{R} \\ (m, z) &\longmapsto m, \end{aligned}$$

see [Gan14]. In other words, $\mathcal{U}(1)^-$ has as objects $\mathbb{R}$ and as arrows

$$\left\{ x \xrightarrow{z} x + m \;\middle|\; x \in \mathbb{R},\, m \in \mathbb{Z}, \text{ and } z \in U(1) \right\},$$

composing two arrows means multiplying their labels, and the strict monoidal structure is given by the respective group structures of objects and arrows.

Lemma 4.1. *The weak Schur cover $\mathcal{C}_n$ of $\boldsymbol{\mu}_n$ can be constructed as the strict categorical group corresponding to the sub-crossed module*

$$\begin{array}{ccc} \mathbb{Z} \times \boldsymbol{\mu}_n & \hookrightarrow & \mathbb{Z} \times U(1) \\ \downarrow{\scriptstyle \boldsymbol{\kappa}} & & \downarrow{\scriptstyle \boldsymbol{\upsilon}} \\ \frac{1}{n}\mathbb{Z} & \hookrightarrow & \mathbb{R}. \end{array}$$

Proof. The circle group $U(1)$ acts by multiplication on the spheres $\mathbb{S}^{2k-1} \subset \mathbb{C}^k$, and on

$$\mathbb{S}^\infty = colim_k \mathbb{S}^{2k-1}.$$

We have

$$\begin{aligned} BU(1) &\simeq \mathbb{S}^\infty / U(1) = \mathbb{C}P^\infty \\ B\boldsymbol{\mu}_n &\simeq \mathbb{S}^\infty / \boldsymbol{\mu}_n = L_n^\infty \end{aligned}$$

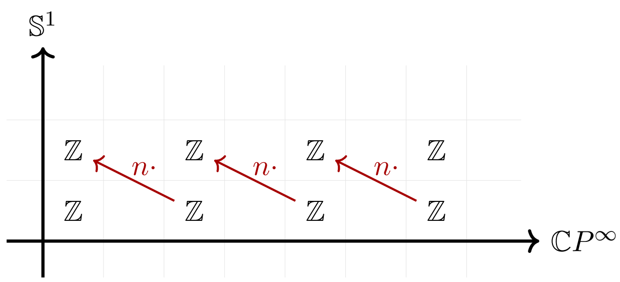
(infinite dimensional lens space). Let $i \colon \boldsymbol{\mu}_n \hookrightarrow U(1)$ be the inclusion map. Then

$$Bi \colon B\boldsymbol{\mu}_n \longrightarrow BU(1)$$

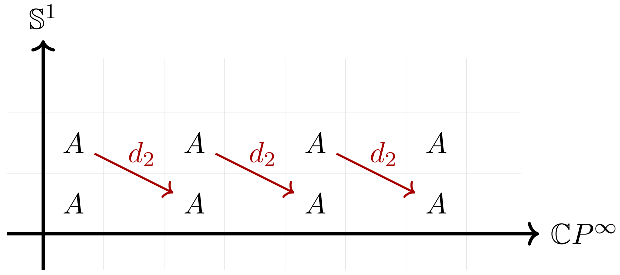
is identified with the quotient map

$$L_n^\infty \longrightarrow \mathbb{C}P^\infty.$$

This is a fibration with fiber $\mathbb{S}^1 \cong U(1)/\boldsymbol{\mu}_n$. Its homology Leray-Serre spectral sequence has E^2-term



For cohomology with coefficients in A, we obtain an E_2-term of the form



with

$$d_2(a) = \underbrace{a + \cdots + a}_{n \text{ times}}.$$

These spectral sequences collapse to give the familiar minimal resolutions

$$\mathbb{Z} \xleftarrow{\;0\;} \mathbb{Z} \xleftarrow{\;n\cdot\;} \mathbb{Z} \xleftarrow{\;0\;} \mathbb{Z} \xleftarrow{\;n\cdot\;} \mathbb{Z} \longleftarrow \cdots$$

and

$$A \xrightarrow{\;0\;} A \xrightarrow{\;n\cdot\;} A \xrightarrow{\;0\;} A \xrightarrow{\;n\cdot\;} A \longrightarrow \cdots$$

We claim that we have a communting diagram

$$\begin{array}{ccccccc}
\mathbb{Z} & = & H^4(\mathbb{C}P^\infty;\mathbb{Z}) & \overset{\sim}{=} & H^3_{gp}(U(1);U(1)) & & \\
\downarrow q & & \downarrow Bi^* & & \downarrow i^* & & \\
\mathbb{Z}/n\mathbb{Z} & = & H^4(L^\infty_n;\mathbb{Z}) & \overset{\sim}{=} & H^3(\boldsymbol{\mu}_n;U(1)) & \xleftarrow{\cong} & H^3(\boldsymbol{\mu}_n;\boldsymbol{\mu}_n) \\
\downarrow \cong & & & & \downarrow \cong & & \downarrow \cong \\
\boldsymbol{\mu}_n & = & & = & Hom(\boldsymbol{\mu}_n, U(1)) & = & Hom(\boldsymbol{\mu}_n;\boldsymbol{\mu}_n),
\end{array}$$

where q is the quotient map, and the equal signs refer to the standard identifications. The commutativite of the top left square follows from the fact that Bi^* is the edge homomorphism in our cohomology spectral sequence. The commutativity of the bottom left square is a diagram chase, involving the minimal resolutions for cohomology with coefficients in $\mathbb{Z}$, $\mathbb{R}$ and $U(1)$. It follows that the restriction of $\mathcal{U}(1)^-$ to $\boldsymbol{\mu}_n$ is a choice of $\mathcal{C}_n[i]$. This categorical group $\mathcal{C}_n[i]$ with center $U(1)$ associated to $\mathcal{C}_n$ determines $\mathcal{C}_n$ up to equivalence. It follows that the categorical group associated to $\boldsymbol{\kappa}$ is a choice of $\mathcal{C}_n$. □

There is an alternative description of the categorical group $\mathcal{C}_n$. For $a \in \frac{1}{n}\mathbb{Z}$, we write

$$a = \lfloor a \rfloor + a',$$

where the Gauß bracket $\lfloor a \rfloor$ denotes the largest integer less than or equal to a.

Definition 4.2 ([HLY14], [JS93, Sec.3, p.49])**.** Let $\mathcal{C}'_n$ be the skelettal 2-group constructed from the $\boldsymbol{\mu}_n$-valued 3-cocycle

$$\alpha(a|b|c) := \exp(\lfloor a' + b' \rfloor c') = \exp(\lfloor a' + b' \rfloor c)$$

on $\frac{1}{n}\mathbb{Z}/\mathbb{Z}$.

Lemma 4.3. *We have an equivalence of 2-groups between $\mathcal{C}'_n$ and $\mathcal{C}_n$.*

Proof. We define a monoidal equivalence from $\mathcal{C}'_n$ to $\mathcal{C}_n$. On objects, we let F be the map

$$\begin{aligned} F\colon \tfrac{1}{n}\mathbb{Z}/\mathbb{Z} &\longrightarrow \tfrac{1}{n}\mathbb{Z} \\ [a] &\longmapsto a', \end{aligned}$$

and on arrows, we let F be the map

$$\begin{aligned} F\colon \left(\tfrac{1}{n}\mathbb{Z}/\mathbb{Z}\right) \times \boldsymbol{\mu}_n &\longrightarrow \tfrac{1}{n}\mathbb{Z} \ltimes (\mathbb{Z} \times \boldsymbol{\mu}_n) \\ ([a], z) &\longmapsto (a', 0, z). \end{aligned}$$

We then define the natural transformation

$$\phi\colon F([a]) + F([b]) \longrightarrow F([a] + [b])$$

given by the arrows

$$(a' + b', -\lfloor a' + b' \rfloor, 1)$$

in $\mathcal{C}_n$. It is elementary to check that (F, ϕ) is indeed a monoidal equivalence. □

5. Platonic 2-groups

The previous section will serve as blueprint for our discussion of the Platonic 2-groups. Let G be a finite subgroup of the three sphere. It is well known[4] that the (co)homology of G is periodic with period 4, with the reduced integral homology concentrated in odd degrees,

$$H_i(G) = \begin{cases} \mathbb{Z} & \text{if } \ i = 0, \\ G^{ab} & \text{if } \ i \equiv 1 \mod 4, \\ \boldsymbol{\mu}_{|G|} & \text{if } \ i \equiv 3 \mod 4, \text{ and} \\ 0 & \text{if } i > 0 \text{ is even,} \end{cases}$$

and the integral cohomology concentrated in even degrees,

$$H^i(G) = \begin{cases} \mathbb{Z} & \text{if } \ i = 0, \\ G^{ab} & \text{if } \ i \equiv 2 \mod 4, \\ \boldsymbol{\mu}_{|G|} & \text{if } \ i \text{ is a positive multiple of 4, and} \\ 0 & \text{else.} \end{cases}$$

In particular, G possesses a weak categorical Schur cover with center $\boldsymbol{\mu}_{|G|}$. The following proposition shows that $\mathcal{G}_{uni}$ can be realized as a sub-categorical group of the third String 2-group.

[4] Periodicity is a theorem by Artin and Tate [AT68], the full statement is a combination of [CE99, XII.2(4),XII.11.1, XVI.9 Application 4]. See also [FHHP04, Cor. 3.1] for a direct proof (following Schur) that the Schur multiplier vanishes and [TZ08] for an explicit resolution and a description of the product structure in cohomology.

Proposition 5.1. *The restriction of* $String(3)$ *to* G *is equivalent to the categorical group with center* $U(1)$ *associated to* $\mathcal{G}_{uni}$ *via the canonical inclusion* $i\colon \boldsymbol{\mu}_{|G|} \hookrightarrow U(1)$,

$$String(3)|_G \simeq \mathcal{G}_{uni}[i].$$

Proof. We follow the argument in the proof of Lemma 4.1, with the difference that the circle group of complex units, $U(1) \subset \mathbb{C}$, is replaced by the three sphere of unit quaternions, $\mathbb{S}^3 \subset \mathbb{H}$. Viewing

$$\mathbb{S}^\infty = colim\, \mathbb{S}^{4n-1}$$

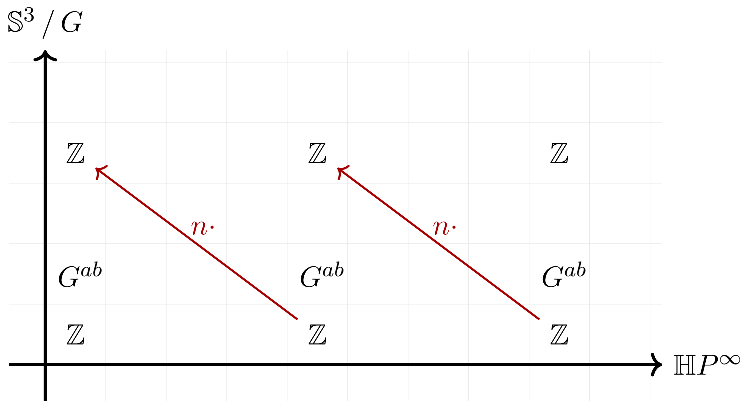
as the colimit of the spheres in $\mathbb{H}^n$, we have

$$\begin{array}{rcccl} B\mathbb{S}^3 & \simeq & \mathbb{S}^\infty / \mathbb{S}^3 & = & \mathbb{H}P^\infty, \qquad \text{and} \\ BG & \simeq & \mathbb{S}^\infty / G. & & \end{array}$$

If j is the inclusion of G in $\mathbb{S}^3$, then Bj becomes the fibration

$$\begin{array}{ccc} \mathbb{S}^3/G & \hookrightarrow & BG \\ & & \downarrow Bj \\ & & \mathbb{H}P^\infty. \end{array}$$

The fibre is the spherical three manifold $\mathbb{S}^3/G$ and, in particular, connected and oriented. We get the following picture of the Leray-Serre spectral sequence for integral homology.



The only non-trivial differential d_4 is multiplication by $n = |G|$. The remainder of the proof is identical to that of Lemma 4.1 □

Remark 5.2. The spherical three manifolds turning up as fibres in the above proof have been the object of intense study. For instance, if G is the binary icosahedral group, then the space $\mathbb{S}^3/G$ is the exotic homology 3-sphere of Poincaré.

Remark 5.3. In the abelian case, where G is a finite cyclic subgroup of $\mathbb{S}^3$, the inclusion j factors through a maximal torus,

$$\begin{array}{ccccc} & & \mathbb{S}^1 & & \\ & {}^{i}\nearrow & & \searrow^{k} & \\ G & & \xrightarrow[j]{} & & \mathbb{S}^3, \end{array}$$

and the commuting diagram

$$
\begin{array}{ccccccc}
 & & H^3_{gp}\left(\mathbb{S}^3; U(1)\right) & \xrightarrow{\;k^*\;} & H^3_{gp}\left(\mathbb{S}^1; U(1)\right) & & \\
 & & \| & & \| & & \\
 & & H^*\left(\mathbb{H}P^\infty; \mathbb{Z}\right) & \xrightarrow{\;(Bk)^*\;} & H^*\left(\mathbb{C}P^\infty; \mathbb{Z}\right) & & \\
 & & \| & & \| & & \\
|v| = 4 & & \mathbb{Z}[\![v]\!] & \longrightarrow & \mathbb{Z}[\![x]\!] & & |x| = 2 \\
 & & v & \longmapsto & x^2 & &
\end{array}
$$

identifies the restriction of $String(3)$ to $\mathbb{S}^1$ with the categorical group $\mathcal{U}(1)^-$ of the previous section.

6. The string covers of the alternating groups

Let S_n be the symmetric group on n elements, and let ϱ_n be its permutation representation. The alternating group $A_n \subset S_n$ is the subgroup of even permutations. We will write $\widetilde{A}_n$ for its spin double cover. In this section, we will introduce a family of categorical groups $\mathscr{A}_n$, fitting into commuting diagrams

$$
\begin{array}{ccc}
\mathbb{B}\pi_3(\mathbb{S}^0) & \xhookrightarrow{\;e\;} & \mathbb{B}U(1) \\
\downarrow & & \downarrow \\
\mathscr{A}_n & \hookrightarrow & String(n) \\
\downarrow & & \downarrow \\
\widetilde{A}_n & \xhookrightarrow{\;\widetilde{\varrho}_n\;} & Spin(n) \\
{\scriptstyle \sigma_n}\downarrow & & \downarrow{\scriptstyle \kappa_n} \\
A_n & \xhookrightarrow{\;\bar{\varrho}_n\;} & SO(n) \\
{\scriptstyle \iota_n}\downarrow & & \downarrow{\scriptstyle \varepsilon_n} \\
S_n & \xhookrightarrow{\;\varrho_n\;} & O(n).
\end{array}
$$

Here e is the Adams e-invariant, the arrows with Greek names are the canonical maps, and in each tower, the top two vertical arrows describe a categorical central extension. These $\mathscr{A}_n$ are characterized, up to equivalence, by

$$\mathscr{A}_n[e] \simeq String(n)|_{\widetilde{A}_n}.$$

Definition 6.1. We will refer to categorical groups $\mathscr{A}_n$ as above as the *string covers of the alternating groups* or simply as the *alternating 2-groups*.

6.1 The Whitehead tower of the plus construction The content of this section is folclore, see for instance the *Mathoverflow* discussion *Plus construction considerations*. Let X be a connected CW-complex with basepoint, whose fundamental group has perfect commutator subgroup

$$P = [\pi_1(X), \pi_1(X)].$$

Let

$$p\colon X \longrightarrow X^+$$

be a homology isomorphism such that

$$p_*(P) = 0 \subseteq \pi_1\left(X^+\right). \tag{5}$$

These conditions are satisfied if and only if the map p is universal, in the homotopy category, with respect to the property (5). This universal property of the plus construction determines X^+ up to unique isomorphism in the homotopy category. We use the notation X^+ whenever the above conditions are satisfied, even when we are working in the strict category. Given p as above, we may pull back the Whitehead tower of X^+ to a tower of fibrations over X,

$$\begin{array}{ccccccccc}
X & = & X_1 & \overset{\xi_1}{\twoheadleftarrow} & X_2 & \overset{\xi_2}{\twoheadleftarrow} & X_3 & \twoheadleftarrow & \cdots \\
\downarrow p & & \downarrow p_1 & & \downarrow p_2 & & \downarrow p_3 & & \\
X^+ & = & W_1 & \twoheadleftarrow & W_2 & \twoheadleftarrow & W_3 & \twoheadleftarrow & \cdots
\end{array}$$

Using the Lerray-Serre spectral sequence, one shows inductively that the p_i are homology isomorphisms. So, the fundamental group of X_i is perfect for $i > 1$, and

$$p_i\colon X_i \longrightarrow W_i$$

satisfies the universal property for its plus construction. In particular, W_i is a choice for X_i^+, and the homology of the tower $X_\bullet$ encodes the homotopy groups of the plus construction of X. More precisely,

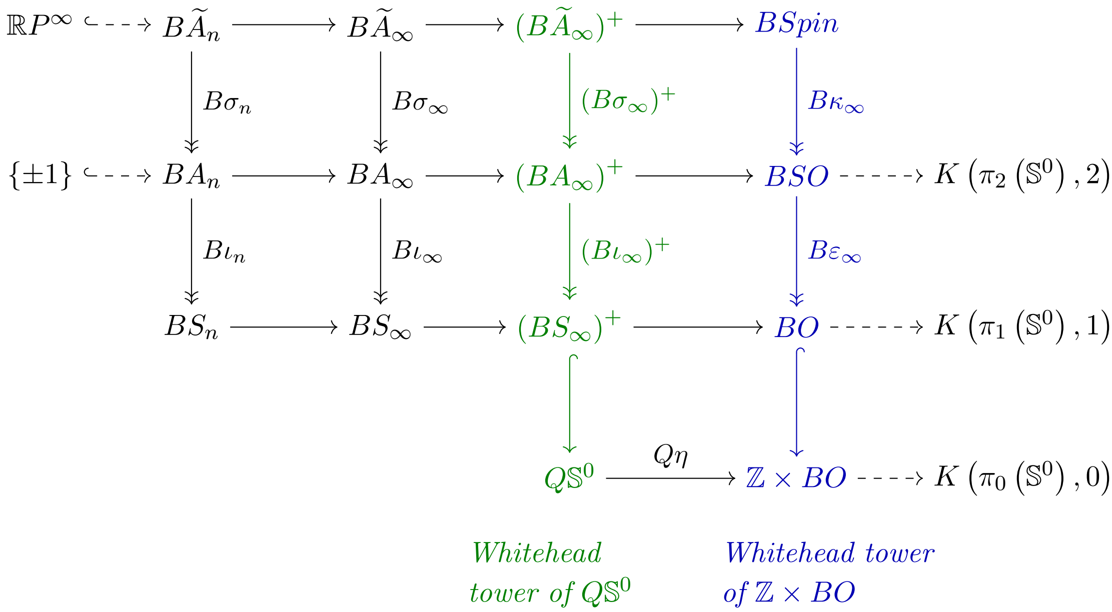
$$\widetilde{H}_i(X_j) = \begin{cases} 0 & \text{if } i < j, \\ \pi_i(X^+) & \text{if } i = j. \end{cases}$$

It is possible to construct the tower $X_\bullet$ directly from X. For this, we let $X_1 = X$ and then proceed inductively, as follows: once X_i has been constructed, let $K(H_i(X_i), i)$ be the ith Eilenberg-MacLane space for the group $H_i(X_i)$, and define ξ_i as the fibration classified by the map

$$f_i\colon X_i \longrightarrow K(H_i(X_i), i)$$

corresponding to $id_{H_i(X_i)}$ under the universal coefficient theorem. We will refer to the resulting tower as the *homology tower* of X. For instance, $X_2 = \widetilde{X}/P$ is the quotient of the universal cover of X by the perfect group P.

Theorem 6.2. *We have a diagram of pull-back squares,*



Here $Q\mathbb{S}^0 = \operatorname{colim} \Omega^n \mathbb{S}^n$ is the infinite loop space of the sphere spectrum, and $\left(Q\mathbb{S}^0\right)_0$ the connected component of its basepoint, while $\eta\colon \mathbb{S}^0 \longrightarrow KO$ is the unit map. The composition of the solid horizontal arrows gives the maps induced, respectively, by the representation ϱ_n and its lifts $\bar{\varrho}_n$ and $\widetilde{\varrho}_n$.

Proof of Theorem 6.2. It is well known that η induces isomorphisms on

$$\pi_0 = \mathbb{Z} \quad \text{and} \quad \pi_1 = \mathbb{Z}/2\mathbb{Z} \quad \text{and} \quad \pi_2 = \mathbb{Z}/2\mathbb{Z}.$$

So, the map $Q\eta$ pulls back the first three steps of the Whitehead tower of $\mathbb{Z} \times BO$ to the first three steps of the Whitehead tower of $Q\mathbb{S}^0$. The Barratt-Quillen-Priddy theorem yields a homology isomorphism

$$p\colon BS_\infty \longrightarrow \left(Q\mathbb{S}^0\right)_0,$$

satisfying

$$p_*(A_\infty) = 0 \qquad \text{and} \qquad (Q\eta)\circ p = B\varrho_\infty.$$

So, the Whitehead tower of $Q\mathbb{S}^0$ is identified with the plus construction of the homology tower of BS_∞. It remains to identify this homology tower in the relevant degrees. The first step is the pull-back of $B\varepsilon_\infty$ along $B\varrho_\infty$. This is the non-trivial double cover $B\iota_\infty$, classified by the map

$$B(sgn)\colon BS_\infty \xrightarrow{\quad B\varrho_\infty \quad} BO \xrightarrow{\quad Bdet \quad} B\{\pm 1\}.$$

Indeed,

$$X_2 = \widetilde{X}/P = ES_\infty/A_\infty.$$

Next, the double cover of $B\varrho_\infty$ is $B\bar{\varrho}_\infty$ and pulls back $B\kappa_\infty$ to the fibration $\xi_2 = B\sigma_\infty$. The classifying map f_2 represents the class

$$[f_2] \in H^2\left(BA_\infty; H_2\left(BA_\infty\right)\right)$$

classifying the Schur cover of A_∞. Finally, the lift of $B\bar{\varrho}_\infty$ to $B\widetilde{A}_\infty$ is $B\widetilde{\varrho}_\infty$. □

As an immediate consequence of the theorem, we obtain half of the tower promised in the introduction as restrictions of the short exact sequence

$$A_\infty \longrightarrow S_\infty \longrightarrow \pi_1(\mathbb{S}^0),$$

the Schur cover of $\widetilde{A}_\infty$

$$\pi_2(\mathbb{S}^0) \longrightarrow \widetilde{A}_\infty \longrightarrow A_\infty,$$

and the categorical Schur cover of the superperfect group $\widetilde{A}_\infty$

$$\mathbb{B}\pi_3(\mathbb{S}^0) \longrightarrow \mathscr{A}_\infty \longrightarrow \widetilde{A}_\infty.$$

In other words, we can now construct the nth alternating 2-group as

$$\mathscr{A}_n := \mathscr{A}_\infty|_{\widetilde{A}_n}.$$

Consider the homomorphisms

$$\begin{array}{rclclcl} b_1\colon H_1(S_n) & \longrightarrow & H_1(S_\infty) & \cong & \pi_1(\mathbb{S}^0) & \cong & \boldsymbol{\mu}_2 \\ b_2\colon H_2(A_n) & \longrightarrow & H_2(A_\infty) & \cong & \pi_2(\mathbb{S}^0) & \cong & \boldsymbol{\mu}_2 \\ b_3\colon H_3(\widetilde{A}_n) & \longrightarrow & H_3(\widetilde{A}_\infty) & \cong & \pi_3(\mathbb{S}^0) & \cong & \boldsymbol{\mu}_{24}, \end{array}$$

where the middle isomorphisms are induced by the Barratt-Priddy-Quillen map.

Lemma 6.3 ([Hau78, 7.2.3])**.** *The map b_1 is an isomorphism for $n \geq 2$, the map b_2 is an isomorphism for $n = 4, 5$ or $n \geq 8$, and the map b_3 is an isomorphism for $n = 4$, $n = 8$ or $n \geq 11$.*

Proof. It is well known that the abelianization of S_n is $\mathbb{Z}/2\mathbb{Z}$ for $n \geq 2$. The values where b_2 is an isomorphism are also well known. This goes back to work of Schur. For $5 \leq n \leq \infty$, the group A_n is perfect. In this range, we have a compatible system of isomorphisms

$$H_2(A_n) \quad \cong \quad \pi_2(BA_n^+).$$

Similarly, we have compatible isomorphisms

$$H_3(\widetilde{A}_n) \quad \cong \quad \pi_3(B\widetilde{A}_n^+),$$

for $n = 5$ and $8 \leq n \leq \infty$. Further, when A_n is perfect, the fibration

$$B\{\pm 1\} \rightarrowtail B\widetilde{A}_n^+ \twoheadrightarrow BA_n^+.$$

[Hau78, Prop.7.1.3] yields an isomorphism

$$\pi_3(B\widetilde{A}_n^+) \quad \cong \quad \pi_3(BA_n^+).$$

Apart from the case $n = 4$, which we will treat in Lemma 6.9, the statement of the Lemma can now be read off from the proof of Proposition A in [Hau78]. □

In low degrees, we still have:

Corollary 6.4. *When A_n is perfect, then its spin extension $\widetilde{A}_n$ is classified by the homomorphism b_2. When $\widetilde{A}_n$ is superperfect, then its string extension $\mathscr{A}_n$ is classified by the homomorphism b_3.*

6.2 The Adams e-invariant Given a ring spectrum E with unit map $\eta = \eta_E$, we may form the exact triangle

$$E[-1] \longrightarrow \overline{E} \longrightarrow \mathbb{S}^0 \xrightarrow{\eta} E$$

in the stable homotopy category.[5] In the case $E = KO$, we have

$$\pi_{4k-1}(KO) = 0 \qquad \text{and} \qquad \pi_{4k}(KO) = \mathbb{Z}.$$

In positive degrees, the stable homotopy groups of spheres are finite. It follows that for $k \geq 1$, the map $\pi_{4k}(\eta)$ is zero, so that we obtain a short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \pi_{4k-1}(\overline{KO}) \longrightarrow \pi_{4k-1}(\mathbb{S}^0) \longrightarrow 0.$$

For a finite abelian group π, we further have the isomorphism

$$Ext(\pi, \mathbb{Z}) \cong Hom(\pi, \mathbb{Q}/\mathbb{Z}) \tag{6}$$

resulting from the injective resolution

$$\mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z}.$$

Definition 6.5. For $k \geq 1$, we let

$$e\colon \pi_{4k-1}(\mathbb{S}^0) \longrightarrow \mathbb{Q}/\mathbb{Z}$$

be the homomorphism classifying the extension $[\pi_{4k-1}(\overline{KO})]$ above.

Lemma 6.6. *Our definition of e agrees with the definition of the Adams e-invariant in* [AS74, (1.1)] *and* [APS75, (4.11)].

Proof. Following the discussion of the complex e-invariant in [CF66, III,16], Atiyah and Smith identify the real e-invariant of a framed manifold M of dimension $4k - 1$ as

$$e(M) = \begin{cases} \widehat{A}(B) & k \text{ even}, \\ \frac{1}{2}\widehat{A}(B) & k \text{ odd}, \end{cases}$$

where B is any spin manifold with boundary $\partial B = M$. They argue that this is a well-defined element of $\mathbb{Q}/\mathbb{Z}$ by the integrality result [AH59, Cor.2(ii)]. To understand this formulation, consider the maps of exact triangles

$$\begin{array}{ccccc}
\mathbb{S}^0 & = & \mathbb{S}^0 & = & \mathbb{S}^0_{\mathbb{Q}} \\
\downarrow \eta & & \downarrow \eta & & \downarrow \\
MSpin & \xrightarrow{\widehat{\mathcal{A}}} & KO & \longrightarrow & KO_{\mathbb{Q}} \\
\downarrow & & \downarrow & & \downarrow \\
\overline{MSpin}[1] & \longrightarrow & \overline{KO}[1] & \longrightarrow & \overline{KO}_{\mathbb{Q}}[1] \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{S}^1 & = & \mathbb{S}^1 & = & \mathbb{S}^1_{\mathbb{Q}},
\end{array}$$

[5] This is the first step in the construction of the E-based Adams-Novikov spectral sequence.

where $\widehat{\mathcal{A}}$ is the Atiyah-Bott-Shapiro orientation [ABS64]. Following [LM89, (7.9),(7.13),(7.17)], this yields a diagram with exact columns

$$\begin{array}{ccccc}
\Omega^{Spin}_{4k} & \xrightarrow{\;ind\;} & \mathbb{Z} & \xrightarrow{\;1\;} & \mathbb{Q} \\
\downarrow & & \downarrow & & \downarrow \cong \\
\Omega^{Spin,fr}_{4k} & \longrightarrow & \pi_{4k-1}(\overline{KO}) & \longrightarrow & \pi_{4k-1}(\overline{KO})_{\mathbb{Q}} \\
\downarrow & & \downarrow & & \downarrow \\
\Omega^{fr}_{4k-1} & = & \pi_{4k-1}(\mathbb{S}^0) & \longrightarrow & 0,
\end{array}$$

where ind is the Atiyah-Milnor-Singer invariant,

$$ind(X) = \begin{cases} \widehat{A}(X) & k \text{ even}, \\ \frac{1}{2}\widehat{A}(X) & k \text{ odd}. \end{cases}$$

We claim that, for even k, the composite of the red arrows sends a spin manifold with framed boundary to the integral over its $\widehat{A}$-class. Indeed, this relative $\widehat{A}$-genus is a homomorphism from $\Omega^{Spin,fr}_{4k}$ to $\mathbb{Q}$, which for closed manifolds agrees with the $\widehat{A}$-genus. Since the inclusion

$$\Omega^{Spin}_{4k} \hookrightarrow \Omega^{Spin,fr}_{4k}$$

becomes an isomorphism after tensoring with $\mathbb{Q}$, this property determines the relative $\widehat{A}$-genus uniquely. By the identical argument, the red arrows compose to half the relative $\widehat{A}$-genus for k odd. We may now reformulate the definition [AS74, (1.1)] as follows: Given an element x of $\pi_{4k-1}(\mathbb{S}^0)$, choose a pre-image $\overline{x}$ of x in $\pi_{4k-1}(\overline{KO})$ and take $e(x)$ to be the image of $\overline{x}$ in

$$\pi_{4k-1}(\overline{KO})_{\mathbb{Q}} \xleftarrow[\cong]{} \pi_{4k}(KO)_{\mathbb{Q}} = \mathbb{Q}$$

modulo

$$\pi_{4k}(KO) = \mathbb{Z}.$$

This description of e coincides with the classifying map of the extension $[\pi_{4k-1}(\overline{KO})]$. □

Lemma 6.7. *The composite*

$$H^4(BSpin;\mathbb{Z}) \xrightarrow{(B\widetilde{\varrho}_\infty)^*} H^4(B\widetilde{A}_\infty;\mathbb{Z}) \overset{\sim}{=} Ext(H_3(B\widetilde{A}_\infty),\mathbb{Z})$$

sends the preferred generator of $H^4(BSpin;\mathbb{Z})$ *to the extension* $[\pi_3\overline{KO}]$ *of*

$$H_3(B\widetilde{A}_\infty) \cong \pi_3(\mathbb{S}^0),$$

used in Definition 6.5.

Proof. The naturality of the universal coefficient theorem (the isomorphism in the lemma) allows us to replace $B\widetilde{A}_\infty$ with $B\widetilde{A}^+_\infty$ and $B\widetilde{\varrho}_\infty$ with $B\widetilde{\varrho}^+_\infty$. Let

$$\xi\colon BSpin \longrightarrow K(\mathbb{Z},4)$$

represent the preferred generator. Then $(B\widetilde{\varrho}^+_\infty)^*([\xi])$ is represented by the composite

$$\xi' = \xi \circ B\widetilde{\varrho}^+_\infty.$$

We have a homotopy commutative diagram

$$\begin{array}{ccccccccc}
\ldots & \longrightarrow & K(\mathbb{Z},3) & \longrightarrow & hofib(\xi') & \longrightarrow & B\widetilde{A}^+_\infty & \xrightarrow{\xi'} & K(\mathbb{Z},4) \\
 & & \Omega\xi\uparrow & & \uparrow & & \| & & \uparrow\xi \\
\ldots & \longrightarrow & Spin & \longrightarrow & hofib(B\widetilde{\varrho}^+_\infty) & \longrightarrow & B\widetilde{A}^+_\infty & \xrightarrow{B\widetilde{\varrho}^+_\infty} & BSpin \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\ldots & \longrightarrow & O & \longrightarrow & Q\overline{KO} & \longrightarrow & Q\mathbb{S}^0 & \xrightarrow{Q\eta} & \mathbb{Z}\times BO,
\end{array}$$

whose rows are homotopy fiber sequences. Using the long exact sequence of (unstable) homotopy groups, we find that all the spaces in the top two rows are 2-connected. In fact, the second row forms the 2-connected cover of the third row. Using Hurwicz and the fact that there are no non-trivial homomorphisms from a finite group to $\mathbb{Z}$, we arrive at the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & H_3(hofib(\xi')) & \longrightarrow & H_3(B\widetilde{A}^+_\infty) & \longrightarrow & 0 \\
 & & \| & & \| & & \| & & \\
0 & \longrightarrow & \pi_3(K(\mathbb{Z},3)) & \longrightarrow & \pi_3(hofib(\xi')) & \longrightarrow & \pi_3(B\widetilde{A}^+_\infty) & \longrightarrow & 0 \\
 & & \| & & \| & & \| & & \\
0 & \longrightarrow & \pi_3(Spin) & \longrightarrow & \pi_3(hofib(B\widetilde{\varrho}^+_\infty)) & \longrightarrow & \pi_3(B\widetilde{A}^+_\infty) & \longrightarrow & 0 \\
 & & \| & & \| & & \| & & \\
0 & \longrightarrow & \pi_3(O) & \longrightarrow & \pi_3(Q\overline{KO}) & \longrightarrow & \pi_3(Q\mathbb{S}^0) & \longrightarrow & 0.
\end{array}$$

The universal coefficient theorem identifies the class $[\xi']$ with the extension on the top row, while the bottom row is the extension in Definition 6.5. □

Corollary 6.8. *The restriction of String(n) to $\widetilde{A}_n$ is equivalent, in a manner unique up to unique isomorphism, to the categorical group with center $U(1)$ associated to $\mathscr{A}_n$ via the Adams e-invariant,*

$$\mathscr{A}_n[e] \simeq String(n)|_{\widetilde{A}_n}.$$

Proof. This follows from the commutativity of the diagram

$$\begin{array}{ccccc}
H^4(BSpin;\mathbb{Z}) & \xrightarrow{(B\widetilde{\varrho}_\infty)^*} & H^4(B\widetilde{A}_\infty;\mathbb{Z}) & \xleftarrow{\cong} & Ext(H_3(B\widetilde{A}_\infty),\mathbb{Z}) \\
\cong\uparrow & & \cong\uparrow & & \uparrow\cong \\
H^3_{gp}(Spin;U(1)) & \xrightarrow{\widetilde{\varrho}^*_\infty} & H^3(\widetilde{A}_\infty;U(1)) & \xrightarrow{\cong} & Hom(H_3(\widetilde{A}_\infty),U(1)),
\end{array}$$

where the horizontal isomorphisms on the right are given by the universal coefficient theorem, and the right-most vertical isomorphism is (6). The left two vertical isomorphisms come from the long exact sequence associated to the short exact sequence of coefficients

$$\mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z}.$$

□

One interpretation of the isomorphism (6) uses the fact that the circle group $U(1)$ is a classifying space for $\mathbb{Z}$. So, the central extensions of π by $\mathbb{Z}$ are classified by homotopy classes of group homomorphisms from π to $U(1)$, and we have

$$K(\mathbb{Z},4) \;=\; B^3U(1).$$

Applying the construction $B(-)^+$ to the categorical central extensions of this section, we obtain the map of exact triangles

$$\begin{array}{ccccccc}
B^2\pi_3(\mathbb{S}^0) & & \overset{B^2e}{\dashrightarrow} & & B^2U(1) & & \\
& \searrow & & & & \searrow & \\
& & B\mathscr{A}_\infty^+ & \xrightarrow{B\mathscr{R}^+} & & & BString \\
& & \downarrow & & & & \downarrow \\
& & B\widetilde{A}_\infty^+ & \xrightarrow{B\widetilde{\varrho}_\infty^+} & & & BSpin \\
& & & \searrow & & & & \searrow \\
& & & & B^3\pi_3(\mathbb{S}^0) & \overset{B^3e}{\dashrightarrow} & & & B^3U(1),
\end{array}$$

whose middle square adds another floor to the map of Whitehead towers in Theorem 6.2. In particular,

$$H_4(B\mathscr{A}_\infty;\mathbb{Z}) \;\cong\; \pi_4(\mathbb{S}^0) \;= 0.$$

Lemma 6.9. *The canonical inclusion of $\widetilde{A}_4$ in $\widetilde{A}_\infty$ induces an isomorphism in degree three homology, sending the fundamental class of the tetrahedral spherical 3-form to the second Hopf map $\nu\colon \mathbb{S}^7 \longrightarrow \mathbb{S}^4$,*

$$\begin{array}{ccc}
H_3(\mathbb{S}^\infty/\widetilde{A}_4) \;=\; H_3(B\widetilde{A}_4) & \xrightarrow{\;\cong\;} & H_3(B\widetilde{A}_\infty) \;\cong\; \pi_3(\mathbb{S}^0) \\
[\mathbb{S}^3/\widetilde{A}_4] & \longmapsto & [\nu].
\end{array}$$

Proof. Applying the plus construction (with respect to $\widetilde{A}_\infty$) to the fibration

$$\begin{array}{ccc}
Spin/\widetilde{A}_\infty & \hookrightarrow & B\widetilde{A}_\infty \\
& & \big\downarrow B\widetilde{\varrho}_\infty \\
& & BSpin,
\end{array}$$

we obtain the identification

$$\left(Spin/\widetilde{A}_\infty\right)^+ \;=\; hofib\left(B\widetilde{\varrho}_\infty^+\right),$$

see [Far96, 3.D.3(2)]. From the proof of Lemma 6.7, we therefore have the short exact sequence

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H_3(Spin) & \longrightarrow & H_3(Spin/\widetilde{A}_\infty) & \longrightarrow & H_3(B\widetilde{A}_\infty) & \longrightarrow & 0 \\
& & \| & & \| & & \| & & \\
0 & \longrightarrow & \mathbb{Z} & \xrightarrow{24\cdot} & \mathbb{Z} & \longrightarrow & \mathbb{Z}/24\mathbb{Z} & \longrightarrow & 0,
\end{array}$$

whose first map can be identified with the differential

$$d_4\colon H_4(BSpin) \longrightarrow H_3(Spin/\widetilde{A}_\infty)$$

in the Leray-Serre spectral sequence for $B\widetilde{\varrho}_\infty$. This can be compared to the scenario for the platonic 2-groups. In particular, we have the commuting diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z} & \xrightarrow{24\cdot} & \mathbb{Z} & \longrightarrow & \mathbb{Z}/24\mathbb{Z} & \longrightarrow & 0 \\
 & & \| & & \| & & \| & & \\
0 & \longrightarrow & H_4(BSpin(3)) & \xrightarrow{d_4} & H_3(Spin(3)/\widetilde{A}_4) & \longrightarrow & H_3(B\widetilde{A}_4) & \longrightarrow & 0 \\
 & & \cong\downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H_4(BSpin) & \xrightarrow{d_4} & H_3(Spin/\widetilde{A}_\infty) & \longrightarrow & H_3(B\widetilde{A}_\infty) & \longrightarrow & 0.
\end{array}$$

Here we are using a non-standard inclusion of $Spin(3)$ inside $Spin(4) \subset Spin$, covering the orthogonal complement of the trivial summand of the permutation representation. This map still gives an isomorphism in H_3, implying that all the vertival arrows are isomorphisms. It follows that the generator of

$$H_3(B\widetilde{A}_\infty) \cong \pi_3(\mathbb{S}^0)$$

with e-invariant $\frac{1}{24}$ is the image of the fundamental class of $Spin(3)/\widetilde{A}_4$ under its inclusion in $B\widetilde{A}_\infty$. □

Corollary 6.10. *The fourth alternating 2-group, $\mathscr{A}_4$, is the weak categorical Schur cover of the binary tetrahedral group, while*

$$\mathscr{A}_3 \simeq \mathcal{C}_6$$

is the weak categorical Schur cover of the cyclic group on six elements.

Remark 6.11. In [FGMNS16], Femina, Galves, Neto and Sreafico describe the fundamental domain of the action of $2T$ on the three sphere as an octahedron (the join of two geodesic segments). This yields a specific description of the fundamental class of $\mathbb{S}^3/2T$. It would be interesting to identify this class with an explicit group cocycle in the spirit of Section 3 or to give a more direct relationship with the second Hopf map.

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